“Struck by Lightning” Supplementary Materials

Exercise about “sticks and lines”

As a group, take one of the “sticks” which is (hopefully) the same length as the width of the classroom’s floor tiles, and one index card. Use the classroom floor as the lined surface, by counting the east-west tile edges as lines (and ignoring the north-south edges).

This exercise concerns the following question. Suppose we toss the stick randomly onto the floor (being careful to avoid furniture and walls, but otherwise being as random as possible), and wait for it to stop. What is the probability, p, that the stick will end up touching one of the (east-west) lines? As a group, consider this question, as follows.

Experimental part:

1. Repeatedly toss the stick. Keep track of the number of times you throw (n), and the number of times the stick ends up touching an east-west line (m). Do this many times, e.g. perhaps 100 times or more.

Data analysis part:

2. Using your data from question 1, what is your best estimate for the probability p?

3. Using your data from question 1, give a range of values such that p will be in that range “19 times out of 20”. [Hint: don’t forget about “margins of error” from polls.]

4. On the index card, write all group members names, together with your values for n and m, your best estimate of p, and your “19 times out of 20” range for p. When you have finished this, give your group’s index card to the instructor.

5. This experiment is somewhat analogous to conducting a public opinion poll, with each stick toss corresponding to phoning a randomly-chosen citizen and asking them (say) whether or not they plan to vote for the governing party. In that analogy:

(a) What would the true value of p correspond to?

(b) Would we know (or be able to figure out) the true value of p? Discuss.

Angles part:

Using careful mathematics, we will now compute p precisely. Let L be the length of the stick (which is equal to the distance between the east-west lines). Let D be the north-south distance from the north-most part of the stick to the next line south. Let H be the north-south distance from the north-most part of the stick to the south-most part of the stick. And, let θ be the angle that the stick’s direction makes with the east-west direction (when moving counter-clockwise starting from directly east).

6. Draw a diagram showing L, D, θ, and some of the lines. (You might wish to draw the diagram twice – once when the stick crosses a line, and once when it doesn't.)

7. Explain why θ is equally likely to be any value between 0 and 180 degrees.

8. Explain why D is equally likely to be any value between 0 and L.

9. Explain why the stick will touch a line if and only if $D \leq H$.

10. Suppose we knew the angle θ at which the stick would end up. Compute H in terms of L and θ. [Hint: Don't forget about trigonometry.]

11. In terms of θ, what is the probability that the stick will end up touching the line? [Hint: combine the last three questions.]

Calculus part:

Since we don't actually know θ, the true probability p is equal to the *average* of the probabilities from the previous question, averaged over all the different possible θ values.

12. **(a)** If you know about integrals (from calculus), then find a way to average over θ, to obtain a formula for the true value of p. [Hint: To compute the average, you have to *integrate* appropriately, and then *divide* by an appropriate value. The final answer involves the quantity "π".]

(b) If you don't know about integrals, then instead estimate this average by computing the average of the probabilities when $\theta = 0$, $\theta = 10$ degrees, $\theta = 20$ degrees, etc. (Then, once you have obtained an estimate, ask the instructor for the actual formula.)

Estimation of π part:

13. Suppose you didn't know the value of π. Re-arrange the formula for p, to get an approximation for π in terms of p.

14. Explain how to estimate π in terms of m and n. [Hint: use the previous question, and recall how you estimated p in terms of m and n.]

15. Use your actual values of n and m (from your earlier experiment) to estimate π. How close was your estimate? [This method of estimating π is called *Buffon's needle*. It was used successfully by Captain O.C. Fox in 1864; see page 188 of the book.]

Law of Large Numbers part:

16. Suppose you were to throw the stick n times for *very large* n, and find that m of those times it was touching the line. What would the Law of Large Numbers tell us in this case, about m/n and about the resulting estimate of π? [This property of estimators is called being *consistent*.]