Supplement for the end of Section 3.5 (page 92) of

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Theorem 3.5.3 raises the question of what distribution the limiting random variable X could have.

If the martingale $\{X_n\}$ is also a *Markov chain*, then we can say more. Call $x \in \mathbf{R}$ an *isolated value* if x is bounded away from all (other) states, i.e. if there is $\delta > 0$ such that $|j - x| \geq \delta$ for all $j \in S$ with $j \neq x$.
(So, if $S \subseteq \mathbf{Z}$, then every state is isolated with $\delta = 1$, and every non-integer $x \in \mathbf{R}$ is isolated with $\delta = \min(x - \lfloor x \rfloor,\ \lceil x \rceil - x) > 0$.)

(3.5.s1) Proposition. If $\{X_n\}$ is a Markov chain on a state space S, which converges w.p. 1 to some random variable X, and $i \in S$ is an isolated value with $p_{ii} < 1$, then $\mathbf{P}(X = i) = 0$.

Proof: If a sequence converges to an isolated value, then it must eventually be constant, i.e. equal to that value an infinite number of times in a row. But if $p_{ii} < 1$, then the probability that $\{X_n\}$ is equal to i infinitely many times in a row is given by $(p_{ii})^\infty = 0$, so $\mathbf{P}(X = i) = 0$. ∎

Combining Proposition 3.5.s1 with Theorem 3.5.3 then gives:

(3.5.s2) Corollary. If $\{X_n\}$ is a Markov chain on a state space S, which is also a martingale, and is bounded below or above, then $\{X_n\}$ converges w.p. 1 to a random variable X having the property that whenever $\mathbf{P}(X = x) > 0$, then either $x \in S$ with $p_{xx} = 1$, or x is not an isolated value.

(3.5.s3) Example. Let $\{X_n\}$ be a Markov chain with state space $S = \{5, 6, 7, 8, \ldots\}$, $p_{5,5} = 1$, $p_{i,i-1} = p_{i,i} = p_{i,i+1} = 1/3$ for $i \geq 6$, and $X_0 = 8$.
Then $\{X_n\}$ is a martingale by (3.1.3), and is bounded below by 5.
Also every value is isolated, and only state $i = 5$ has $p_{ii} = 1$.
Hence, by (3.5.s2), $\{X_n\}$ converges w.p. 1 to a random variable X, such that $\mathbf{P}(X = x) = 0$ for all $x \neq 5$. Hence, $\mathbf{P}(X = 5) = 1$.
So, $\{X_n\}$ converges w.p. 1 to the constant 5, i.e. $X_n \to 5$. ∎

Corollary 3.5.s2 can then be used in the solutions to Problems 3.5.4(b), 3.5.5(b), and 3.5.6(f).