Gambling Games and Random Walks

Jeffrey S. Rosenthal, Department of Statistics, University of Toronto

PRESENTATION PART.

Set-Up 1:

Suppose you start with a dollars, and repeatedly bet one dollar until you either reach 0 dollars (i.e. go broke), or reach c dollars (i.e. get rich), and then you stop. Suppose you have probability $1/2$ of winning (or losing) each bet.

Or equivalently: Suppose a frog starts at lily pad number a, and repeatedly jumps one lily pad to the left or right, until they reach pad number 0 or pad number c, and then they stop. Suppose the frog has probability $1/2$ of jumping either left or right each time.

QUESTION: What is the probability q of reaching c before 0?

SOLUTION METHOD #1 (outline): Let $s(a)$ be this probability. Then $s(a) = (1/2)\,s(a+1) + (1/2)\,s(a-1)$, whenever $0 < a < c$. (Why?) Also $s(0) = 0$ and $s(c) = 1$. This is a system of $c+1$ equations in the $c+1$ unknowns $s(0), s(1), \ldots, s(c)$. Re-arranging, we see that $s(a+1) - s(a) = s(a) - s(a-1)$ for $0 < a < c$. It follows that $s(a) = Ka$ for some constant K. Since $s(c) = 1$, we have $K = 1/c$, so $s(a) = a/c$. Hence, $q = s(a) = a/c$.

SOLUTION METHOD #2: Since on average we break even, the amounts of money we have form a *martingale*, i.e. a random sequence which stays the same on average. It follows (!) that our average (or "expected") amount at the end should equal the amount we started with. That is $q(c) + (1-q)(0) = a$, so that $q = a/c$.

Set-Up 2:

Let $0 < p < 1$.

Suppose you start with a dollars, and repeatedly bet one dollar until you either reach 0 dollars (i.e. go broke), or reach c dollars (i.e. get rich). and then you stop. Suppose you have probability p of winning (and probability $1-p$ of losing) each bet.

Or equivalently: Suppose a frog starts at lily pad number a, and repeatedly jumps one lily pad to the left or right, until they reach pad number 0 or pad number c, and then they stop. Suppose the frog has probability p of jumping right (and probability $1-p$ of jumping left) each time.

QUESTION: What is the probability q of reaching c before 0?

If $p = 1/2$ it's the same as set-up 1.

FACT: If $p \neq 1/2$, then

$$q = \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^c - 1} \,. \qquad (*)$$

DISCUSSION TOPICS:

1. (a) Fill in all the details for Solution Method #1 (for Set-Up 1) above.

(b) Modify Solution Method #1 appropriately, to apply it to Set-Up 2. See if you can derive the formula $(*)$.

2. For Set-Up 2, let X_n be the amount of money you have at time n (so that $X_0 = a$, and either $X_1 = a + 1$ or $X_1 = a - 1$, etc.). Let

$$Y_n = \left(\frac{1-p}{p}\right)^{X_n} .$$

(a) Show that the sequence $Y_0, Y_1, \ldots$ is a martingale, i.e. that it stays the same on average.

(b) Conclude that the average value of Y_n once we've reached either c or 0, is equal to $((1-p)/p)^a$.

(c) Use this to solve for q in Set-Up 2. See if you can derive the formula $(*)$.

3. For Set-Up 2, with $p \neq 1/2$, again let X_n be the amount of money you have at time n. Let let J_n be the number of bets (or frog jumps) made up to time n. Let $Z_n = X_n - (2p-1)J_n$.

(a) Show that the sequence $Z_0, Z_1, \ldots$ is a martingale, i.e. that it stays the same on average.

(b) Conclude that the average value of Z_n once we've reached either c or 0, is equal to a.

(c) Use this, together with the formula $(*)$, to solve for the average number of bets (or jumps) that will be made before reaching 0 or c.