STA 2111 (Graduate Probability I), Fall 2025

Homework #1 Assignment: worth 10% of final course grade.

Due: Thursday Sept 25, either: **(a)** Hardcopy in class by 2:10 PM **sharp**, or **(b)** electronic submission (link to be posted on course web page) by 1:45 PM **sharp**, both in Toronto time.

AT THE TOP:

- Please include your full name and usual nickname (if you have one) and student number and department and program and year.

GENERAL NOTES:

- Homework assignments are to be solved by each student individually. You may discuss questions in general terms with other students, and look up general topics in books and internet. But you must solve the problems on your own, and do all of your own writing, without any assistance from other students nor from any AI/chatbot/chatGPT/etc software.

- You should provide very complete solutions, **EXPLAINING ALL REASONING** very clearly. Make your homework neat and readable, e.g. typeset in latex or printed clearly.

- **Late penalty:** 1–5 minutes late is -5%; 5–15 minutes late is -10%; otherwise if x days late then $-20\% \times \text{ceiling}(x)$. So, please don't be late!

THE ACTUAL ASSIGNMENT:

1. Let $\Omega = \{1,2,3,4\}$, and let $\mathcal{J} = \Big\{\emptyset, \{1\}, \{2\}, \{3,4\}, \Omega\Big\}$. Define $\mathbf{P} : \mathcal{J} \to [0,1]$ by $\mathbf{P}(\emptyset) = 0$, $\mathbf{P}\{1\} = 1/7$, $\mathbf{P}\{2\} = 2/7$, $\mathbf{P}\{3,4\} = 4/7$, and $\mathbf{P}(\Omega) = 1$.

(a) [**3**] Prove that $\mathcal{J}$ is a semi-algebra.

(b) [**4**] Find $\mathbf{P}^*(A)$ and $\mathbf{P}^*(A^C)$, where $A = \{2,3\} \subseteq \Omega$ and $\mathbf{P}^*$ is outer measure.

(c) [**4**] Determine whether or not $A \in \mathcal{M}$, where $\mathcal{M}$ is the σ-algebra constructed in the proof of the Extension Theorem. [Hint: Perhaps consider the case $E = \Omega$.]

2. [**5**] Prove that the extension $(\Omega, \mathcal{M}, \mathbf{P}^*)$ constructed in the proof of the Extension Theorem must be "complete", i.e. if $A \in \mathcal{M}$ with $\mathbf{P}^*(A) = 0$, and $B \subseteq A$, then $B \in \mathcal{M}$.

3. Let $\Omega = \{1,2,3,4\}$, and $\mathcal{F} = 2^\Omega$ the collection of all subsets of Ω, and

$$\mathcal{C} \;=\; \Big\{\emptyset, \{1,2\}, \{2,3\}, \{3,4\}, \Omega\Big\}.$$

Define functions $\mu, \nu : \mathcal{F} \to [0,1]$ by $\mu(A) = \frac{1}{2}\mathbf{1}_A(1) + \frac{1}{2}\mathbf{1}_A(3)$ and $\nu(A) = \frac{1}{2}\mathbf{1}_A(2) + \frac{1}{2}\mathbf{1}_A(4)$, where e.g. $\mathbf{1}_A(3) = 1$ if $3 \in A$ or $\mathbf{1}_A(3) = 0$ if $3 \notin A$.

(a) [**5**] Prove that $(\Omega, \mathcal{F}, \mu)$ is a valid probability triple. (It then follows similarly that $(\Omega, \mathcal{F}, \nu)$ is also a valid probability triple.)

(b) [**3**] Determine whether $\mathcal{C}$ is an algebra.

(c) **[3]** Determine whether $\mathcal{C}$ is a semi-algebra.

(d) **[5]** Find $\sigma(\mathcal{C})$, the smallest σ-algebra containing all elements of $\mathcal{C}$.

(e) **[3]** Determine whether $\mu(A) = \nu(A)$ for all $A \in \mathcal{C}$.

(f) **[3]** Determine whether $\mu(A) = \nu(A)$ for all $A \in \mathcal{F}$.

(g) **[3]** Explain why these facts do not contradict our theorem about uniqueness of extensions of probability measures.

4. For any interval $I \subseteq [0, 1]$, let $\mathbf{P}(I)$ be the length of I.

(a) **[5]** Prove that if $I_1, I_2, \ldots, I_n$ is a finite collection of intervals, and if $\bigcup_{j=1}^{n} I_j \supseteq I_*$ for some interval I_*, then $\sum_{j=1}^{n} \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: Suppose I_j has left endpoint a_j and right endpoint b_j, and first re-order the intervals so $a_1 \leq a_2 \leq \ldots \leq a_n$.]

(b) **[5]** Prove that if $I_1, I_2, \ldots$ is a countable collection of open intervals, and if $\bigcup_{j=1}^{\infty} I_j \supseteq I_*$ for some closed interval I_*, then $\sum_{j=1}^{\infty} \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: You may use the Heine-Borel Theorem, which says that if a collection of open intervals contain a closed interval, then some finite sub-collection of the open intervals also contains the closed interval.]

(c) **[5]** Prove that if $I_1, I_2, \ldots$ is *any* countable collection of intervals, and if $\bigcup_{j=1}^{\infty} I_j \supseteq I_*$ for *any* interval I_*, then $\sum_{j=1}^{\infty} \mathbf{P}(I_j) \geq \mathbf{P}(I_*)$. [Hint: Extend the interval I_j by $\epsilon\, 2^{-j}$ at each end, and decrease I_* by ϵ at each end, while making I_j open and I_* closed. Then use part (b).] (Note: This is the "countable monotonicity" property needed to apply the Extension Theorem for the Uniform[0,1] distribution, to guarantee that $\mathbf{P}^*(I) \geq \mathbf{P}(I)$.)

(d) **[4]** Suppose we instead defined $\mathbf{P}(I)$ to be the square of the length of I. Show that in that case, the conclusion of part (c) would not hold.

[END; total points = 60]