STA 2111 (Graduate Probability I), Fall 2024

Homework #2 Assignment: worth 10% of final course grade.

Due: in class by 2:10 p.m. sharp (Toronto time) on Thursday Nov. 14.

GENERAL NOTES:

- Homework assignments are to be solved by each student individually. You may discuss questions in general terms with other students, but you must solve them on your own, including doing all of your own computing and writing.

- You should provide very complete solutions, including explaining all of your reasoning clearly. Please submit your assignment as hard copy at the beginning of class.

- **Late penalty:** 1–5 minutes late is -5%; 5–15 minutes late is -10%; otherwise if x days late then $-20\% \times \text{ceiling}(x)$. So, don't be late!

THE ACTUAL ASSIGNMENT:

1. [**5**] Let $A_1, A_2, \ldots$ be independent events. Let Y be a random variable which is measurable with respect to $\sigma(A_n, A_{n+1}, \ldots)$ for each $n \in \mathbf{N}$. Prove there is $a \in \mathbf{R}$ with $\mathbf{P}(Y = a) = 1$. [Hints: You may wish to consider $\mathbf{P}(Y \le y)$. And don't forget Kolmogorov.]

2. Let X be a non-negative random variable with $\mathbf{P}(X > 0) > 0$.

(a) [**3**] Prove that there must exist some $n \in \mathbf{N}$ such that $\mathbf{P}(X \ge 1/n) > 0$.

(b) [**3**] Prove that we must have $\mathbf{E}(X) > 0$.

3. For each of the following conditions, either give an example of a random variable X defined on Lebesgue measure on $[0, 1]$ satisfying those conditions, or prove that no such random variable exists.

(a) [**3**] $X \ge 0$, and $\mathbf{E}(X) = 3$, and $\mathbf{P}(X \ge 7) = 1/2$.

(b) [**3**] $\mathbf{E}(X) = 3$, and $\mathbf{P}(X \ge 7) = 1/2$.

(c) [**3**] $\mathbf{E}(X) = 3$, and $\mathbf{Var}(X) = 4$, and $\mathbf{P}(X \le 0) = 1/2$.

4. Let $\{X_n\}$ and X be random variables defined on Lebesgue measure on $[0, 1]$, with $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for each fixed $\omega \in [0, 1]$. Suppose for each $n \in \mathbf{N}$, $\mathbf{E}|X_n| < \infty$, and also $X_{n+1}(\omega) \le X_n(\omega)$ for all $\omega \le 1/2$, and $X_{n+1}(\omega) \ge X_n(\omega)$ for all $\omega > 1/2$.

(a) [**4**] Prove that $\lim_{n\to\infty} \mathbf{E}(X_n \mathbf{1}_{[0,\frac{1}{2}]}) = \mathbf{E}(X \mathbf{1}_{[0,\frac{1}{2}]})$. [Hint: What can you say about $Y_n = X_1 \mathbf{1}_{[0,\frac{1}{2}]} - X_n \mathbf{1}_{[0,\frac{1}{2}]}$? And, remember that $\mathbf{E}|X_1| < \infty$.]

(b) [**4**] Prove that $\lim_{n\to\infty} \mathbf{E}(X_n \mathbf{1}_{(\frac{1}{2},1]}) = \mathbf{E}(X \mathbf{1}_{(\frac{1}{2},1]})$.

(c) [**4**] Assuming $\mathbf{E}|X| < \infty$, prove that $\lim_{n\to\infty} \mathbf{E}(X_n) = \mathbf{E}(X)$.

(d) [**4**] Without assuming that $\mathbf{E}|X| < \infty$, find an example of such $\{X_n\}$ where $\mathbf{E}(X)$ is undefined, and hence $\lim_{n\to\infty} \mathbf{E}(X_n) \ne \mathbf{E}(X)$.

5. Let Y be <u>any non-negative</u> random variable.

(a) [**3**] Prove that $Y\,\mathbf{1}_{Y>0} = Y$. [Hint: Consider separately $Y(\omega) = 0$ and $Y(\omega) > 0$.]

(b) [**5**] Assuming that $0 < \mathbf{E}(Y^2) < \infty$, prove that $\mathbf{P}(Y > 0) \geq (\mathbf{E}(Y))^2 / \mathbf{E}(Y^2)$. [Hints: apply Cauchy-Schwarz with $X = \mathbf{1}_{Y>0}$. And don't forget part (a).]

6. Suppose X and $\{X_n\}$ are all random variables defined on some $(\Omega, \mathcal{F}, \mathbf{P})$ where Ω is <u>countable</u> and $\mathcal{F} = 2^{\Omega}$, and that $\{X_n\} \to X$ in probability.

(a) [**4**] Prove that if $\omega \in \Omega$ with $\mathbf{P}(\{\omega\}) > 0$, then for all $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $\mathbf{P}(|X_n - X| \geq \epsilon) < \mathbf{P}(\{\omega\})$ for all $n \geq N$.

(b) [**4**] Prove that $\{X_n\} \to X$ almost surely. [Hint: Part (a) might help.]

7. Let $\{I_j\}_{j=1}^{\infty}$ be independent random variables, with $I_1 \sim \text{Uniform}\{1, 2, \ldots, 9\}$, $I_2 \sim \text{Uniform}\{10, 11, 12, \ldots, 99\}$, $I_3 \sim \text{Uniform}\{100, 101, 102, \ldots, 999\}$, and in general, I_j is chosen uniformly from the set of all positive integers having j digits.
Then, define $\{X_n\}_{n=1}^{\infty}$ by: $X_n = 7$ if $n = I_j$ for some j, otherwise $X_n = 5$.

(a) [**4**] <u>Prove or disprove</u> that $\{X_n\} \to 5$ in probability.

(b) [**4**] <u>Prove or disprove</u> that $\{X_n\} \to 5$ almost surely.

[END; total points = 60]