STA 2111 (Graduate Probability I), Fall 2022

Homework #2 Assignment: worth 10% of final course grade.

Due: in class by 10:10 a.m. sharp (Toronto time) on Thursday Nov. 24.

GENERAL NOTES:

- Homework assignments are to be solved by each student individually. You may discuss questions in general terms with other students, but you must solve them on your own, including doing all of your own computing and writing.

- You should provide very complete solutions, including explaining all of your reasoning clearly. Please submit your assignment as hard copy in class.

- **Late penalty:** 1–5 minutes late is −5%; 5–15 minutes late is −10%; otherwise if x days late then $-20\% \times \text{ceiling}(x)$. So, don't be late!

THE ACTUAL ASSIGNMENT:

1. Let Ω be a finite non-empty set, and let $\mathcal{J}$ consist of all singletons in Ω, together with $\emptyset$ and Ω. Let $f : \Omega \to [0,1]$ with $\sum_{\omega \in \Omega} f(\omega) = 1$, and define $\mathbf{P}(\emptyset) = 0$, $\mathbf{P}(\Omega) = 1$, and $\mathbf{P}\{\omega\} = f(\omega)$ for all $\omega \in \Omega$.

(a) [**3**] Prove that $\mathcal{J}$ is a semialgebra.

(b) [**3**] Compute $\mathbf{P}^*(A)$ for any $A \subseteq \Omega$, where P^* is outer measure.

(c) [**3**] Describe precisely the collection $\mathcal{M}$ (as defined in the Extension Theorem).

2. Let $\mathbf{P}$ and $\mathbf{Q}$ be two probability measures on the same space Ω and σ-algebra $\mathcal{F}$.

(a) [**3**] Suppose that $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{F}$ with $\mathbf{P}(A) \le \frac{1}{2}$. Prove that $\mathbf{P} = \mathbf{Q}$, i.e. that $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{F}$.

(b) [**3**] Give an example where $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{F}$ with $\mathbf{P}(A) < \frac{1}{2}$, but such that $\mathbf{P} \ne \mathbf{Q}$, i.e. that $\mathbf{P}(A) \ne \mathbf{Q}(A)$ for some $A \in \mathcal{F}$.

3. Let $(\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$ be Lebesgue measure on $[0,1]$. Consider a second probability triple, $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$, defined as follows: $\Omega_2 = \{1,2\}$, $\mathcal{F}_2$ consists of all subsets of Ω_2, and $\mathbf{P}_2$ is defined by $\mathbf{P}_2\{1\} = \frac{1}{3}$, $\mathbf{P}_2\{2\} = \frac{2}{3}$, and additivity. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the product measure of $(\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$.

(a) [**5**] Express each of Ω, $\mathcal{F}$, and $\mathbf{P}$ as explicitly as possible.

(b) [**3**] Find a set $A \in \mathcal{F}$ such that $\mathbf{P}(A) = \frac{3}{4}$.

4. [**6**] Let $([0,1]^2, \mathcal{F}, \lambda)$ be Lebesgue measure on $[0,1]^2$, i.e. the product measure $\text{Unif}[0,1] \times \text{Unif}[0,1]$. Let A be the triangle $\{(x,y) \in [0,1]^2 \,;\, y < x\}$. Prove $A \in \mathcal{F}$, and compute $\lambda(A)$.

(Continued on other side.)

5. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability triple, let $B, C \in \mathcal{F}$ be two fixed events, and let

$$A_n = \begin{cases} B, & n \text{ odd} \\ C, & n \text{ even} \end{cases}$$

In terms of B and C:

(a) [**3**] Specify the events $\liminf_n A_n$ and $\limsup_n A_n$.

(b) [**3**] Specify the values $\liminf_{n\to\infty} \mathbf{P}(A_n)$ and $\limsup_{n\to\infty} \mathbf{P}(A_n)$.

(c) [**3**] Show directly why

$$\mathbf{P}\Big(\liminf_n A_n\Big) \;\le\; \liminf_{n\to\infty} \mathbf{P}(A_n) \;\le\; \limsup_{n\to\infty} \mathbf{P}(A_n) \;\le\; \mathbf{P}\Big(\limsup_n A_n\Big).$$

6. [**5**] Let $A_1, A_2, \ldots$ be independent events. Let Y be a random variable which is measurable with respect to $\sigma(A_n, A_{n+1}, \ldots)$ for each $n \in \mathbf{N}$. Prove there is $a \in \mathbf{R}$ with $\mathbf{P}(Y = a) = 1$.

7. Give examples of events A, B, and C, each with probability not 0 or 1, such that:

(a) [**4**] $\mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B)$, $\mathbf{P}(A \cap C) = \mathbf{P}(A)\,\mathbf{P}(C)$, and $\mathbf{P}(B \cap C) = \mathbf{P}(B)\,\mathbf{P}(C)$, but it is not the case that $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\,\mathbf{P}(B)\,\mathbf{P}(C)$. [Hint: You can let Ω be a set of four equally likely points.]

(b) [**4**] $\mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B)$, $\mathbf{P}(A \cap C) = \mathbf{P}(A)\,\mathbf{P}(C)$, and $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\,\mathbf{P}(B)\,\mathbf{P}(C)$, but it is not the case that $\mathbf{P}(B \cap C) = \mathbf{P}(B)\,\mathbf{P}(C)$. [Hint: You can let Ω be a set of eight equally likely points.]

8. Let X be a non-negative random variable with $\mathbf{P}(X > 0) > 0$.

(a) [**4**] Prove that there exists some $\delta > 0$ such that $\mathbf{P}(X \ge \delta) > 0$.

(b) [**4**] Prove that $\mathbf{E}(X) > 0$.

9. Let $(\Omega, \mathcal{F}, P)$ be Lebesgue measure on $[0, 1]$, and set

$$X(\omega) \;=\; \begin{cases} 2\,, & \omega \text{ rational} \\ 3\,, & \omega = 1/\sqrt{3} \\ 9\,, & \text{other } 0 \le \omega < 1/5 \\ 4\,, & \text{other } 1/5 \le \omega < 3/5 \\ 7\,, & \text{other } 3/5 \le \omega \le 1\,. \end{cases}$$

(a) [**3**] Compute $\mathbf{P}(X < 6)$.

(b) [**3**] Compute $\mathbf{E}(X)$.

10. [**4**] Let X be a non-negative random variable with finite mean, and let $a \in \mathbf{R}$ be any real number. Prove that $\mathbf{E}\big[\max(X, a)\big] \ge \max\big[\mathbf{E}(X), a\big]$.

[END; total points = 69]