STA 198F, Fall 2020: Probabilities Everywhere

Class activities for Week 3.

Class Poll Plan: Which question? How to sample?

Whole Class discussion: As a group, we will discuss last week's homework readings and questions. Be sure to participate actively, and raise your hand often!

Last Week's "Gambler's Ruin" Guesses, by Breakout Room number:
1: 21%. 2: 30.5%. 3: 40%. 4: 14.3%. 5: 12.5%. 6: 30%. 7: 14.3%. 8: 26.33%.

"Gambler's Ruin" Mathematical Solution. (On next two pages.)

Homework assignment (upload to Quercus by 2:30 PM before the next class):

Read the beginning of Chapter 3, from page 23 to page 30 (stop at "Another game is keno"), and also pages 41 to 43 (the section "A Life of Large Numbers"). While you read, consider and make notes about the following questions:

1. What are the "two key facts" which explain why casinos always make money?

2. What is the Law of Large Numbers, and how is it related to casino profits?

3. Summarise the "Honorable Bus Fine" story. What can we conclude from it?

4. What are all the different roulette bets which are considered? For each of those bets, how is the average outcome calculated? What does it work out to be?

5. What "gambler's ruin" question is considered (p. 27)? What answer is provided? How does this answer compare with what you would have guessed? How difficult do you think it would be to compute this answer mathematically?

6. Summarise the "Slowly but Surely" story. What can we conclude from it?

7. Summarise at least three of the examples presented in the section "A Life of Large Numbers". Do you find the asserted connections to the Law of Large Numbers to be convincing? Why or why not?

8. Describe at least two examples from your own life, of events which are somehow related to the Law of Large Numbers.

STA 198F, Fall 2020: Probabilities Everywhere

Math Exercise: Solving the Gambler's Ruin Problem

Recall the rules of our game: **A** starts with six chips; **B** starts with two chips. They repeatedly make bets. On each bet, with probability 1/3, **B** gives one chip to **A**; or, with probability 2/3, **A** gives one chip to **B**. The game continues until one person wins all eight chips; that person is the "winner". We will now figure out, mathematically, the probability that **A** wins the game. (Note that this is not an easy problem, and it will take us a number of weeks and many steps to solve.)

1. What is the probability that **A** wins the game on the second bet?

2. What is the probability that **A** wins the game on the third bet? On the fifth bet? On any odd numbered bet?

3. What is the probability that **A** wins the game on the fourth bet? [This is a bit harder, you have to consider several different cases.]

4. What is the probability that **A** wins the game on the sixth bet? [This is still harder, there are more cases to consider.]

5. Do you think it would be possible to continue in this way, for the eighth bet, the tenth bet, etc., and eventually find the precise total probability that **A** wins the game?

To make progress solving this problem, we will think about the problem in a new way. As a first step, we will let $s(a)$ stand for the probability that **A** wins this game if they start with a chips (and **B** starts with $8-a$ chips). Thus, $s(6)$ is the probability that we actually want to find, while e.g. $s(3)$ is the probability that **A** wins this game if instead they start with 3 chips and **B** starts with 5 chips.

6. Make sure everybody in your group understands this notation. For example, what does $s(5)$ stand for? What about $s(4)$?

7. Since all we really want to know is $s(6)$, why do you think we need to introduce all these other numbers $s(a)$ for different values of a?

8. Why is it necessary to use mathematical notation for these probabilities? For example, why is it better to write $s(6)$ than to write out "the probability that **A** wins this game if they start with 6 chips" each time?

9. Although the probabilities $s(a)$ are difficult to compute for most values of a, there are two probabilities which are very easy to compute, namely $s(0)$ and $s(8)$. What does $s(0)$ equal? What does $s(8)$ equal?

10. Make your *best guess* as to the values of $s(a)$ for all $a = 0, 1, 2, 3, 4, 5, 6, 7, 8$.
[**Note:** you can't actually solve for the $s(a)$ values yet; just guess values for them.]

We next relate these different unknown probabilities $s(a)$ to each other.

11. Find a formula for $s(6)$ in terms of $s(5)$ and $s(7)$. That is, suppose we knew both $s(5)$ and $s(7)$. Could we then easily compute $s(6)$? By what formula? [**Hint:** Suppose **A** starts with six chips. Consider the two possibilities for what happens after the first bet. How many chips will **A** have then?] **Note:** This question is the key to the whole exercise. It will require careful thought, though the resulting formula is not complicated.

12. Similarly, find a formula for $s(3)$ in terms of $s(2)$ and $s(4)$. Also find a formula for $s(1)$ in terms of $s(0)$ and $s(2)$.

13. More generally, find a formula for $s(a)$ in terms of $s(a-1)$ and $s(a+1)$, for any integer a between 1 and 7 inclusive.

At this point, we have seven unknowns, namely $s(1)$, $s(2)$, $s(3)$, $s(4)$, $s(5)$, $s(6)$, and $s(7)$. We also have seven equations, namely the seven equations from the previous question (one for each different value of "a").

14. Do you think we now have enough information to solve the problem?

You could now perhaps solve the problem on your own. However, the resulting equations are still quite messy. So, you may wish to use the following steps.

15. Find a formula for $s(a+1)$ in terms of $s(a)$ and $s(a-1)$.

16. Now let $x = s(1)$, an unknown quantity. In terms of x, what is $s(2)$? What is $s(3)$? What is $s(4)$? What is $s(8)$?

17. Since we know what $s(8)$ equals, therefore what does x have to be?

18. Putting all of this together, solve for $s(a)$ for all a between 1 and 7. (If possible, try to obtain a simple, general formula for $s(a)$.)

19. Finally, what is the probability $s(6)$, that we wanted to find?

20. How does the true value of $s(6)$ compare to your guess a few classes ago? Do you find the true value surprising?

21. How do your guesses of the other values of $s(a)$ (from question 10 of the original Gambler's Ruin exercise) compare to the true values?

22. (If time.) How would the solution be modified if each bet is for two chips instead of one? Or, if there are a total of "N" chips instead of "8"? Or, if player **A** has probability "p" instead of "1/3" of winning each bet? Can you verify the figure of $1/37,650$ on page 27 of the textbook? Do you find it surprising?