STA447/2006 Midterm #2, March 19, 2020

(ONLINE; 3 hours; 8 questions; 4 pages; total points = 50)

[SOLUTIONS]

1. [**5**] Let $S = \{1, 2, 3, 4\}$, and $\pi_1 = 1/6$, $\pi_2 = 1/12$, and $\pi_3 = 1/2$, $\pi_4 = 1/4$. Find explicit transition probabilities $\{p_{ij}\}_{i,j\in S}$ for a Markov chain on S, with $p_{ij} = 0$ whenever $|i-j| \geq 2$, such that (with proof) $\lim\limits_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$. [Hint: Don't forget Metropolis.]

Solution. *The Metropolis algorithm says the chain will be reversible with respect to π if $p_{i,i+1} = \frac{1}{2}\min[1, \frac{\pi_{i+1}}{\pi_i}]$ and $p_{i,i-1} = \frac{1}{2}\min[1, \frac{\pi_{i-1}}{\pi_i}]$ and $p_{i,i} = 1-p_{i,i+1}-p_{i,i-1}$. Thus, explicitly, $p_{2,1} = p_{2,3} = p_{4,3} = 1/2$, and $p_{1,2} = \frac{1}{2}[(1/12)/(1/6)] = 1/4$, and $p_{3,2} = \frac{1}{2}[(1/12)/(1/2)] = 1/12$, and $p_{3,4} = \frac{1}{2}[(1/4)/(1/2)] = 1/4$. Then $p_{1,1} = 1 - p_{1,2} = 1 - (1/4) = 3/4$, $p_{2,2} = 1 - p_{2,1} - p_{2,3} = 1 - (1/2) - (1/2) = 0$, $p_{3,3} = 1 - p_{3,2} - p_{3,4} = 1 - (1/12) - (1/4) = 2/3$, and $p_{4,4} = 1 - p_{4,3} = 1 - (1/2) = 1/2$. In matrix form,*

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/12 & 2/3 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Then $p_{ij} = 0$ whenever $|i-j| \geq 2$. And P is reversible with respect to π since it is a Metropolis algorithm, so π is stationary. Also, the chain is irreducible since it is possible to go $1 \to 2 \to 3 \to 4$ and $4 \to 3 \to 2 \to 1$. And the chain is aperiodic since e.g. $p_{1,1} > 0$. So, by the Markov Chain Convergence Theorem, $\lim\limits_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$.

2. [**6**] Consider a graph with vertex set $V = \{1, 2, 3, 4, 5\}$, and edge weights $w(1,2) = w(2,1) = w(1,3) = w(3,1) = w(2,3) = w(3,2) = w(1,4) = w(4,1) = w(1,5) = w(5,1) = w(4,5) = w(5,4) = 1$, and $w(u,v) = 0$ otherwise. Let $\{X_n\}$ be random walk on this graph, with $X_0 = 1$. For each of the following limits, determine whether or not the limit exists, and if yes then what it equals: **(a)** $\lim\limits_{n\to\infty} \mathbf{P}(X_n = 1)$, **(b)** $\lim\limits_{n\to\infty} \frac{1}{2}[\mathbf{P}(X_n = 1) + \mathbf{P}(X_{n+1} = 1)]$, and **(c)** $\lim\limits_{n\to\infty} \frac{1}{3}[\mathbf{P}(X_n = 1) + \mathbf{P}(X_{n+1} = 1) + \mathbf{P}(X_{n+2} = 1)]$.

Solution. *Here $Z = \sum_{u,v\in V} w(u,v) = 12 < \infty$, so the walk has stationary distribution given by $\pi_u = d(u)/12$ where $d(u)$ is the degree of vertex $u \in V$. The graph is connected since e.g. it is possible to go $1 \to 2 \to 3 \to 1 \to 4 \to 5 \to 1$, so the random walk is irreducible. And, the walk is aperiodic (i.e., the graph is not bipartite) since e.g. it is possible to go $1 \to 2 \to 1$ and also $1 \to 2 \to 3 \to 1$ so the period of state 1 is $gcd\{2,3\} = 1$. Hence, by the Graph Convergence Theorem, $\lim_{n\to\infty} p_{11}^{(n)} = d(1)/12 = 4/12 = 1/3$. So, $\lim_{n\to\infty} \mathbf{P}_1(X_n = 1) = 1/3$. Hence also $\lim_{n\to\infty} \mathbf{P}_1(X_{n+1} = 1) = 1/3$ and $\lim_{n\to\infty} \mathbf{P}_1(X_{n+2} = 1) = 1/3$. Therefore, limit (a) equals 1/3, and limit (b) equals $\frac{1}{2}[(1/3) + (1/3)] = 1/3$, and limit (c) equals $\frac{1}{3}[(1/3) + (1/3) + (1/3)] = 1/3$, i.e. all three limits exist and all equal 1/3.*

3. [**8**] Suppose each car that passes is independently equally likely to be Red, Green, or Blue. Let τ be the number of cars which pass until the first time we see the pattern Red-Green (i.e., until a Green car passes right after a Red car). Compute $z = \mathbf{E}(\tau)$.

Solution. *Let X_n be the amount of the pattern "RG" that we have achieved after the n^{th} car (starting over as soon as we complete it). Then $\{X_n\}$ is a Markov chain on $S = \{0, 1, 2\}$. Then starting from state 0 or 2 (if we have not yet completed any part of a new RG pattern), the transition probabilities are given by $p_{01} = p_{21} = 1/3$ (if the next car is Red, beginning a new RG pattern), or $p_{00} = p_{20} = 2/3$ (if the next car is Green or Blue, not starting a pattern). And, starting from state 1 (if we have just seen a Red car), then $p_{12} = 1/3$ (if the next car is Green, finishing the pattern), and $p_{11} = 1/3$ (if the next car is Red, restarting the pattern at R), and $p_{10} = 1/3$ (if the next car is Blue, ruining the pattern). That is,*

$$P = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}.$$

Its stationary distribution π must satisfy that $\pi P = \pi$, i.e. $\pi_0 p_{0j} + \pi_1 p_{1j} + \pi_2 p_{2j} = \pi_j$ for all $j \in S$. Setting $j = 0$ gives $\pi_0(2/3) + \pi_1(1/3) + \pi_2(2/3) = \pi_0$. Setting $j = 1$ gives $\pi_0(1/3) + \pi_1(1/3) + \pi_2(1/3) = \pi_1$, and since $\pi_0 + \pi_1 + \pi_2 = 1$ this means that $\pi_1 = 1/3$. Setting $j = 2$ gives $\pi_1(1/3) = \pi_2$, i.e. $\pi_2 = \pi_1(1/3) = 1/9$. Since $\pi_0 + \pi_1 + \pi_2 = 1$ we must have $\pi_0 = 1 - \pi_1 - \pi_2 = 1 - (1/3) - 1/9 = 5/9$. But z is the expected time to go from 0 to 2, or equivalently the mean recurrence time of the state 2. Hence, by the Recurrence Time Theorem, $z = 1/\pi_2 = 1/(1/9) = 9$.

4. [**4**] In the previous question, suppose each Red car is worth 12, each Green car is worth 6, and each Blue car is worth 3. Let Y be the total worth of all cars up to and including time τ. Compute $\mathbf{E}(Y)$. [Note: If you could not solve the previous question, then you may leave your answer to this question in terms of the unknown value "z".]

Solution. *Here τ is a stopping time with finite mean z. And, Y is a sum of i.i.d. car values having mean $m = (1/3)(12) + (1/3)(6) + (1/3)(3) = 4 + 2 + 1 = 7$. Hence, by Wald's Theorem, $\mathbf{E}(Y) = m\,\mathbf{E}(\tau) = 7z = 7(9) = 63$.*

5. [**6**] Find Markov chain transitions $\{p_{ij}\}_{i,j \in S}$ on the state space $S = \{1, 2, 5\}$ (so there are only <u>3 states</u>), such that $p_{21} = 1/2$, and also the chain is a <u>martingale</u>.

Solution. *To be a martingale, we need $\sum_{j \in S} j\, p_{ij} = i$ for each $i \in S$. Setting $i = 2$ and recalling that $p_{21} = 1/2$, we have $(1)(1/2) + (2)p_{22} + (5)p_{25} = 2$. But we must have $p_{21} + p_{22} + p_{25} = 1$, i.e. $(1/2) + p_{22} + p_{25} = 1$, i.e. $p_{22} + p_{25} = 1 - (1/2) = 1/2$, i.e. $p_{22} = (1/2) - p_{25}$. So, $(1)(1/2) + (2)[(1/2) - p_{25}] + (5)p_{25} = 2$, i.e. $(3/2) + 3p_{25} = 2$, i.e. $3p_{25} = 1/2$, i.e. $p_{25} = 1/6$. Then $p_{22} = 1 - p_{21} - p_{25} = 1 - (1/2) - (1/6) = 1/3$. Also, since the states 1 and 5 are endpoints, the only way to stay the same on average is to never move at all, so we must have $p_{11} = p_{55} = 1$, whence $p_{12} = p_{15} = p_{51} = p_{52} = 0$. In matrix form,*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/3 & 1/6 \\ 0 & 0 & 1 \end{pmatrix}.$$

6. [**6**] Consider a Markov chain $\{X_n\}$ with state space $S = \{5, 6, 7, 8, \ldots\}$, with $p_{5,5} = 1$, and $p_{i,i-1} = p_{i,i} = p_{i,i+1} = 1/3$ for all $i \geq 6$, and with $X_0 = 8$. (You may assume that $\mathbf{E}|X_n| < \infty$

for each n.) Then **(a)** show that as $n \to \infty$, the values $\{X_n\}$ converge with probability 1 to some random variable X, and **(b)** determine whether or not $\mathbf{E}(X) = \lim_{n\to\infty} \mathbf{E}(X_n)$.

Solution. *First of all,* $p_{55} = 1$*, and for each* $i \geq 6$ *we have* $\sum_j j\, p_{ij} = (i-1)(1/3) + (i)(1/3) + (i+1)(1/3) = i[(1/3) + (1/3) + (1/3)] - (1/3) + (1/3) = i[1] + 0 = i$*. Hence (since also* $\mathbf{E}|X_n| < \infty$*),* $\{X_n\}$ *is a martingale.*

Then, since $\{X_n\}$ *is a martingale which is non-negative and hence bounded on one side, by the Martingale Convergence Theorem the values* $\{X_n\}$ *must converge with probability 1 to some random variable* X*.*

However, since S *is discrete,* $\{X_n\}$ *can only converge to states* i *with* $p_{ii} = 1$*, which only holds for* $i = 5$*. So, we must have* $\mathbf{P}(X = 5) = 1$*. Hence,* $\mathbf{E}(X) = (5)(1) + 0 = 5$*.*

But since $\{X_n\}$ *is a martingale, therefore* $\mathbf{E}(X_n) = \mathbf{E}(X_0) = 8$ *for each fixed* $n \in \mathbf{N}$*.*

Therefore, $\lim_{n\to\infty} \mathbf{E}(X_n) = \lim_{n\to\infty}(8) = 8 \neq 5 = \mathbf{E}(X)$*, so no we do not have* $\lim_{n\to\infty} \mathbf{E}(X_n) = \mathbf{E}(X)$*.*

7. [9] Consider a branching process with initial value $X_0 = 2$, and with offspring distribution given by $\mu\{0\} = 1/3$ and $\mu\{2\} = 2/3$. Let q be the probability of eventual extinction. Then **(a)** compute $\mathbf{P}(X_1 = 2)$, and **(b)** compute $\mathbf{P}(X_2 = 8)$, and **(c)** determine (with explanation) whether $q = 0$, or $q = 1$, or $0 < q < 1$.

Solution. *For (a), since* $X_0 = 2$*, we could have* $X_1 = 2$ *if either the first individual has 2 offspring (probability 2/3) and the second individual has 0 offspring (probability 1/3), or the first individual has 0 offspring (probability 1/3) and the second individual has 2 offspring (probability 2/3). Hence,* $\mathbf{P}(X_1 = 2) = (2/3)(1/3) + (1/3)(2/3) = 4/9$*.*

For (b), since $X_0 = 2$*, the only way to get* $X_2 = 8$ *is if each of the 2 individuals at time 0 has 2 offspring (probability 2/3 each), and then each of the 4 individuals at time 1 has 2 offspring (probability 2/3 each). Hence,* $\mathbf{P}(X_2 = 8) = (2/3)^2(2/3)^4 = 2^6/3^6 = 64/729$*.*

For (c), we first compute that the reproductive number m *is the mean of* μ*, so* $m = (0)(1/3) + (2)(2/3) = 4/3$*. Since* $m > 1$ *and* $X_0 > 0$*, we know from class that flourishing is possible, and hence extinction is not certain, i.e.* $q < 1$*. On the other hand, the probability that both offspring at time 0 have 0 offspring is* $(\mu\{0\})^2 = (1/3)^2 = 1/9 > 0$*, so extinction is possible, i.e.* $q > 0$*. Hence, we have* $0 < q < 1$*.*

8. [6] Suppose a stock price X_0 at time 0 is equal to 10, and at time S is random with $\mathbf{P}(X_S = 4) = 3/5$ and $\mathbf{P}(X_S = 12) = 2/5$. Suppose there is also an option to buy one share of the stock at time S for price $K = 7$, and this option has been priced (by some company) at the value \$2. Assume you are allowed to buy or sell any amount of the stock at its given price of 10, and also to buy or sell any amount of this option at its listed price of 2. Determine (with explanation) an explicit amount of stock and of this option that you could buy or sell at time 0 to achieve arbitrage, i.e. to make a guaranteed positive profit.

Solution. *Suppose that at time 0, you buy* x *stock shares for \$10 each, and* y *option shares for \$2 each. Then if the stock price goes up to \$12, then you make* $\$12 - \$10 = \$2$ *on each stock share, and make* $\$(12 - 7 - 2) = \$3$ *on each option share, for a total profit of* $2x + 3y$*. But if the stock instead goes down to \$4, you lose* $\$(10 - 4) = \$6$ *on each stock share, and lose \$2 on each option share, for a total profit of* $-6x - 2y$*. These two profits are equal if* $2x + 3y = -6x - 2y$*, i.e.* $8x = -5y$*, i.e.* $y = (-8/5)x$*. For example, if we buy* $x = -5$

shares of the stock (i.e. sell 5 shares), and buy $y = 8$ *shares of the option, then if* $X_S = 12$ *then our profit is* $2(-5) + 3(8) = -10 + 24 = 14 > 0$*, while if* $X_S = 4$ *then our profit is* $(-6)(-5) + (-2)(8) = 30 - 16 = 14 > 0$*, so in either case we make a positive profit, i.e. we have achieved arbitrage.*

[END OF EXAMINATION; total points = 50]