STA447/2006 Midterm #2, March 21, 2019

(135 minutes; 9 questions; 7 pages; total points = 60)

FAMILY NAME: ______________________ **GIVEN NAME(S):** ______________________

STUDENT #: ______________________ **SIGNATURE:** ______________________

CLASS (circle one): STA447 STA2006

- **Do not open this booklet until told to do so. Answer all questions.**
- **Aids allowed: NONE. You may use results from class, with explanation.**
- **Point values** for each question are indicated in [**square brackets**].
- It is important to **explain** all of your solutions clearly.
- You may continue on the **back** of the page if necessary (write "**OVER**").
- **Scrap paper is included at the end of this test (and may be detached).**

DO NOT WRITE BELOW THIS LINE.

Question	Score
1	/5
2	/5
3(a)	/2
3(b)	/3
3(c)	/4
4	/5
5	/4
6(a)	/4
6(b)	/2
6(c)	/3
6(d)	/3

Question	Score
7(a)	/2
7(b)	/2
7(c)	/2
7(d)	/2
8(a)	/3
8(b)	/3
9(a)	/3
9(b)	/3
TOTAL:	/60

1. **[5]** Let $S = \{1,2,3,4\}$, with $\pi_1 = 1/8$, $\pi_2 = 3/8$, and $\pi_3 = \pi_4 = 1/4$. Find (with proof) transition probabilities $\{p_{ij}\}_{i,j \in S}$ for a Markov chain on S, such that $p_{ij} = 0$ whenever $|i-j| \geq 2$, and $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for all $i,j \in S$.

2. **[5]** Consider the Markov chain with state space $S = \{1,2,3,4\}$, $\nu_3 = 1$, and transition probabilities specified by $p_{11} = p_{22} = 1$, $p_{31} = p_{32} = p_{33} = p_{34} = 1/4$, and $p_{42} = p_{43} = p_{44} = 1/3$. Compute $\mathbf{P}_3(T_1 < T_2)$. [Hint: Don't forget how we solved Gambler's Ruin.]

3. Consider a graph with vertex set $V = \{1, 2, 3, 4\}$, and edge weights $w(1, 2) = w(2, 1) = 2$, $w(1, 3) = w(3, 1) = 3$, $w(1, 4) = w(4, 1) = 4$, and $w(u, v) = 0$ otherwise. Let $\{X_n\}$ be random walk on this graph, with $X_0 = 1$.

(a) [**2**] Compute (with explanation) $\mathbf{P}(X_1 = 4)$.

(b) [**3**] Compute (with explanation) $\mathbf{P}(X_3 = 4)$.

(c) [**4**] For each of **(i)** $\lim_{n\to\infty} \mathbf{P}(X_n = 4)$, and **(ii)** $\lim_{n\to\infty} \frac{1}{2}[\mathbf{P}(X_n = 4) + \mathbf{P}(X_{n+1} = 4)]$, determine whether or not the limit exists, and if yes then what it equals.

4. [5] Suppose we repeatedly roll a fair six-sided die (which is equally likely to show 1, 2, 3, 4, 5, or 6). Let τ be the number of rolls until we see 5 twice in a row, i.e. until the pattern "55" first appears. Let $z = \mathbf{E}(\tau)$. Compute z.

5. [4] In the previous question, let X be the sum of all the numbers up to but not including the first "55", and let Y be the sum of all the numbers up to and including the first "55". Compute $\mathbf{E}(X)$ and $\mathbf{E}(Y)$. [Note: If you could not solve the previous question, then you may leave your answers to this question in terms of the unknown value z.]

6. Let $\{X_n\}$ be a Markov chain on the state space $S = \{1, 2, 3, 4\}$, with $X_0 = 3$, and with transition probabilities $p_{11} = p_{44} = 1$, $p_{21} = 1/4$, $p_{34} = 1/5$, and $p_{24} = p_{31} = p_{12} = p_{13} = p_{14} = p_{41} = p_{42} = p_{43} = 0$. Let $T = \inf\{n \geq 0 : X_n = 1 \text{ or } 4\}$, and let $U = T - 1$.

(a) [**4**] Find valid values of p_{22}, p_{23}, p_{32}, and p_{33}, which make $\{X_n\}$ a martingale.

(b) [**2**] For the values found in part (a), compute $\mathbf{E}(X_T)$.

(c) [**3**] For the values found in part (a), compute $p = \mathbf{P}(X_T = 4)$.

(d) [**3**] For the values found in part (a), compute $\mathbf{E}(X_U)$.

7. Consider a Markov chain $\{X_n\}$ with state space $S = \{0, 1, 2, 3, \ldots\}$, with $p_{0,0} = 1$, and $p_{i,0} = p_{i,2i} = 1/2$ for all $i \geq 1$, and with $X_0 = 5$. Let $T = \inf\{n \geq 1 : X_n = 0\}$.

(a) [**2**] Determine whether or not $\{X_n\}$ is a martingale.

(b) [**2**] Determine whether or not $\mathbf{E}(X_n) = 5$ for each fixed $n \in \mathbf{N}$.

(c) [**2**] Determine whether or not $\mathbf{P}(T < \infty) = 1$.

(d) [**2**] Determine whether or not $\mathbf{E}(X_T) = 5$.

8. Let $\{B_t\}_{t\geq 0}$ be standard Brownian motion, and let $\tau = \inf\{t > 0 : B_t = -2 \text{ or } 3\}$.

(a) **[3]** Compute $\mathbf{E}[(2 + B_2 + B_3)^2]$.

(b) **[3]** Compute $p = \mathbf{P}[B_\tau = 3]$.

9. Suppose cars arrive according to a Poisson process with rate $\lambda = 3$ cars per minute, and each car is independently either Blue with probability 1/2, or Green with probability 1/3, or Red with probability 1/6.

(a) **[3]** Let S be the arrival time of the first car that arrives after at least 5 minutes (so we must have $S > 5$). Compute (with explanation) the expected value $\mathbf{E}(S)$.

(b) **[3]** Compute (with explanation) the probability that, in the first 2 minutes, exactly 2 Blue and 1 Green cars arrive.

[END OF EXAMINATION; total points = 60]

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