STA447/2006 Midterm #1, February 7, 2019

(135 minutes; 4 questions; 3 pages; total points = 50)

[SOLUTIONS]

1. Consider a Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities $p_{12} = 1/2$, $p_{13} = 1/2$, $p_{21} = 1/3$, $p_{23} = 2/3$, and $p_{31} = 1$, otherwise $p_{ij} = 0$.

(a) **[2]** Compute $p_{11}^{(2)}$.

Solution. $p_{11}^{(2)} = \sum_{j \in S} p_{1j} p_{j1} = p_{11}p_{11} + p_{12}p_{21} + p_{13}p_{31} = (0)(0) + (1/2)(1/3) + (1/2)(1) = (1/6) + (1/2) = 4/6 = 2/3$.

(b) **[5]** Find a probability distribution π which is stationary for this chain.

Solution. *We need* $\pi P = \pi$, *i.e.* $\sum_{i \in S} \pi_i p_{ij} = \pi_j$ *for all* $j \in S$. *When* $j = 1$ *this gives* $\pi_2(1/3) + \pi_3(1) = \pi_1$, *so* $\pi_3 = \pi_1 - \pi_2(1/3)$. *When* $j = 2$ *this gives* $\pi_1(1/2) = \pi_2$, *so* $\pi_1 = 2\pi_2$. *Combining the two equations,* $\pi_3 = 2\pi_2 - \pi_2(1/3) = (5/3)\pi_2$. *We need* $\pi_1 + \pi_2 + \pi_3 = 1$, *i.e.* $2\pi_2 + \pi_2 + (5/3)\pi_2 = 1$, *i.e.* $(14/3)\pi_2 = 1$. *So,* $\pi_2 = 3/14$. *Then* $\pi_1 = 2\pi_2 = 6/14 = 3/7$, *and* $\pi_3 = (5/3)\pi_2 = (5/3)(3/14) = 5/14$. *As a check, when* $j = 3$ *we need* $\pi_1(1/2) + \pi_2(2/3) = \pi_3$, *i.e.* $\pi_1(1/2) + \pi_2(2/3) = \pi_3$, *i.e.* $(3/7)(1/2) + (3/14)(2/3) = (5/14)$, *i.e.* $(3/14) + (2/14) = (5/14)$, *which also holds. So, the stationary distribution is* $\pi = (3/7, 3/14, 5/14)$.

(c) **[3]** Determine if the chain is reversible with respect to π.

Solution. *No, it is not, since e.g.* $\pi_1 p_{13} = (3/7)(1/2) = 3/14$, *while* $\pi_3 p_{31} = (5/14)(1) = 5/14$, *so* $\pi_i p_{ij} \neq \pi_j p_{ji}$ *in this case.*

(d) **[6]** Determine (with explanation) which of the following statements are true and which are false: (i) $\lim_{n\to\infty} p_{13}^{(n)} = \pi_3$. (ii) $\lim_{n\to\infty} \frac{1}{2}[p_{13}^{(n)} + p_{13}^{(n+1)}] = \pi_3$. (iii) $\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} p_{13}^{(\ell)} = \pi_3$.

Solution. *Here* π *is stationary by part (b), and the chain is irreducible since e.g. it can go* $1 \to 2 \to 3 \to 1$, *and the chain is aperiodic since e.g. it can get from 1 to 1 in two steps* $(1 \to 2 \to 1)$ *or three steps* $(1 \to 2 \to 3 \to 1)$ *and* $gcd(2, 3) = 1$. *Hence, by the Markov chain Convergence Theorem,* $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ *for all* $i, j \in S$, *so (i) holds. Then* $\lim_{n\to\infty} \frac{1}{2}[p_{13}^{(n)} + p_{13}^{(n+1)}] = \frac{1}{2}[\lim_{n\to\infty} p_{13}^{(n)} + \lim_{n\to\infty} p_{13}^{(n+1)}] = \frac{1}{2}[\pi_3 + \pi_3] = \pi_3$, *so (ii) holds. And also, by Average Probability Convergence (or the theory of Cesàro sums),* $\lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} p_{13}^{(\ell)} = \pi_3$, *so (iii) also holds. In summary, all three statements are true.*

(e) **[3]** Determine (with explanation) whether or not $\sum_{n=1}^{\infty} p_{13}^{(n)} = \infty$.

Solution. *Yes it does. The chain is irreducible since e.g. it can go* $1 \to 2 \to 3 \to 1$. *And* $|S| = 3$ *which is finite. So, by the Finite Space Theorem,* $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ *for all* i *and* j, *including when* $i = 1$ *and* $j = 3$.

2. Consider a Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition matrix:

$$P = \begin{pmatrix} 1/4 & 1/2 & 1/8 & 1/8 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \\ 0 & 4/5 & 0 & 1/5 \end{pmatrix}$$

(a) [**4**] Specify (with explanation) which states are recurrent, and which are transient.

Solution. *State 1 is transient, since from 1 with probability 3/4 it leaves 1 immediately and never returns, so* $f_{11} = 1/4 < 1$. *State 3 is recurrent, since from 3 it always stays at 3, so* $f_{33} = 1$. *States 2 and 4 are recurrent, since* $C = \{2, 4\}$ *is a closed finite subset on which the chain is irreducible, so* $f_{ij} = 1$ *for all* $i, j \in C$, *so* $f_{22} = f_{44} = 1$. *Alternatively, use geometric series:* $f_{22} = (1/3) + (2/3)(4/5) + (2/3)(1/5)(4/5) + (2/3)(1/5)^2(4/5) + \ldots = (1/3) + (2/3)(4/5)/[1 - (1/5)] = (1/3) + (2/3)(4/5)/(4/5) = (1/3) + (2/3) = 1$, *and similarly* $f_{44} = (1/5) + (4/5)(2/3) + (4/5)(1/3)(2/3) + (4/5)(1/3)^2(2/3) + \ldots = (1/5) + (4/5)(2/3)/[1 - (1/3)] = (1/5) + (4/5)(2/3)/(2/3) = (1/5) + (4/5) = 1$.

(b) [**3**] Compute f_{24}.

Solution. *Again,* $C = \{2, 4\}$ *is a closed finite subset on which the chain is irreducible, so* $f_{ij} = 1$ *for all* $i, j \in C$, *so* $f_{24} = 1$. *Alternatively, use a geometric series:* $f_{24} = (2/3) + (1/3)(2/3) + (1/3)^2(2/3) + \ldots = (2/3)/[1 - (1/3)] = (2/3)/(2/3) = 1$.

(c) [**3**] Compute f_{14}.

Solution. *When the chain leaves state 1, if it jumps to 3 then it will never hit 4, or if it jumps to 4 then it will of course hit state 4, or if it jumps to 2 then it will eventually hit state 4 since* $f_{24} = 1$ *from the previous part. So,* f_{14} *is the probability that the chain jumps to 2 or 4 [probability 5/8] when it leaves 1 [probability 3/4], i.e.* $f_{14} = \mathbf{P}_1(X_1 = 2 \text{ or } 4 \,|\, X_1 \neq 1) = [(1/2) + 1/8)]/(3/4) = (5/8)/(3/4) = 5/6$. *Alternatively, use geometric series:* $f_{14} = \mathbf{P}_1(\text{eventually hit 2 or 4}) = (5/8) + (1/4)(5/8) + (1/4)^2(5/8) + \ldots = (5/8)/[1 - (1/4)] = (5/8)/(3/4) = 5/6$.

(d) [**3**] Determine whether or not $\sum_{n=1}^{\infty} p_{24}^{(n)} = \infty$.

Solution. *Yes. Again,* $C = \{2, 4\}$ *is a closed finite subset on which the chain is irreducible, so* $\sum_{n=1}^{\infty} p_{ij} = \infty$ *for all* $i, j \in C$, *so yes* $\sum_{n=1}^{\infty} p_{24} = \infty$.

(e) [**3**] Determine whether or not $\sum_{n=1}^{\infty} p_{14}^{(n)} = \infty$.

Solution. *Yes. For example, note that by Chapman-Kolmogorov,* $p_{14}^{(n+1)} \geq p_{12} p_{24}^{(n)} = (1/2) p_{24}^{(n)}$. *So,* $\sum_{n=1}^{\infty} p_{14}^{(n)} \geq \sum_{n=1}^{\infty} p_{14}^{(n+1)} \geq (1/2) \sum_{n=1}^{\infty} p_{24}^{(n)} = (1/2)(\infty) = \infty$, *i.e.* $\sum_{n=1}^{\infty} p_{14}^{(n)} = \infty$.

3. For each of the following sets of conditions, either provide (with explanation) an example of a state space S and Markov chain transition probabilities $\{p_{ij}\}_{i,j \in S}$ such that the conditions are satisfied, or prove that no such a Markov chain exists.

(a) **[3]** There is $k \in S$ having period 1, and $\ell \in S$ having period 3.

Solution. *Yes, possible. For example, let* $S = \{1, 2, 3, 4\}$, *with* $p_{11} = p_{23} = p_{34} = p_{42} = 1$, *and* $p_{ij} = 0$ *otherwise. Then state* $k = 1$ *has period 1 since it returns to 1 immediately, but state* $\ell = 2$ *has period 3 since it only returns in multiples of 3 steps (by* $2 \to 3 \to 4 \to 2$*). (Of course, this chain is not* <u>*irreducible*</u>*; for irreducible chains, all states have the same period by the Equal Periods Lemma.)*

(b) **[3]** The chain is irreducible, and there are distinct states $i, j, k, \ell \in S$ such that $f_{ij} = 1$, and $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} < \infty$.

Solution. *Yes, possible. For example, simple random walk with* $p > 1/2$ *is irreducible, and as shown in class (using the Law of Large Numbers) it has* $f_{ij} = 1$ *for all* $i < j$ *(e.g.* $i = 0$ *and* $j = 5$*), but it is transient so by the Transience Equivalences Theorem* $\sum_{n=1}^{\infty} p_{k\ell}^{(n)} < \infty$ *for all* $k, \ell \in S$ *(e.g.* $k = 2$ *and* $\ell = 4$*).*

(c) **[3]** There are distinct states $i, j, k \in S$ with $f_{ij} = 1/3$, $f_{jk} = 1/4$, and $f_{ik} = 1/20$.

Solution. *No, not possible. One way to eventually get from* i *to* k*, is to first eventually get from* i *to* j*, and then eventually get from* j *to* k*. This means we must have* $f_{ik} \geq f_{ij}\, f_{jk} = (1/3)(1/4) = 1/12 > 1/20$*, so we cannot have* $f_{ik} = 1/20$*.*

4. **[6]** Prove the Equal Periods Lemma, i.e. prove that if $i \leftrightarrow j$, and t_i is the period of state i, and t_j is the period of state j, then $t_i = t_j$. [Note: You cannot <u>use</u> the Equal Periods Lemma or any later results from class to prove this, you have to prove it yourself.]

Solution. *Since* $i \leftrightarrow j$*, we can find* $r, s \in \mathbf{N}$ *with* $p_{ij}^{(r)} > 0$ *and* $p_{ji}^{(s)} > 0$*. Then by Chapman-Kolmogorov,* $p_{ii}^{(r+s)} \geq p_{ij}^{(r)} p_{ji}^{(s)} > 0$*, so* t_i *divides* $r + s$*. Also if* $p_{jj}^{(n)} > 0$*, then* $p_{ii}^{(r+n+s)} \geq p_{ij}^{(r)} p_{jj}^{(n)} p_{ji}^{(s)} > 0$*, so* t_i *divides* $r + n + s$*, so* t_i *divides* n*. Hence,* t_i *is a common divisor of* $\{n \geq 1 : p_{jj}^{(n)} > 0\}$*. Since* t_j *is the* <u>*greatest*</u> *such divisor, therefore* $t_j \geq t_i$*. Exchanging* i *and* j *shows that also* $t_i \geq t_j$*. Hence,* $t_i = t_j$*.*

[END OF EXAMINATION; total points = 50]