# STA447/2006 (Stochastic Processes), Winter 2017

## Homework #1

**(6 questions; 2 pages; total points = 70)**

**Due:** In class by 6:10 p.m. **sharp** on Thursday February 2. **Warning:** Late homeworks, even by one minute, will be penalised (as discussed on the course web page).

**Note:** You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Directly copying other solutions is strictly prohibited!

[Point values are indicated in square brackets. It is very important to **EXPLAIN** all your solutions very clearly – correct answers poorly explained will **NOT** receive full marks.]

**Include at the top of the first page:** Your name and student number, and whether you are enrolled in STA447 or STA2006.

**1.** Consider a (discrete-time) Markov chain $\{X_n\}$ on the state space $S = \{1, 2, 3, 4\}$, with transition probabilities given by

$$(p_{ij}) = \begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 2/5 & 3/5 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

**(a)** [**5**] Compute (with explanation) $p_{43}^{(3)} \equiv \mathbf{P}(X_3 = 3 \,|\, X_0 = 4)$. **Note:** You should compute this exactly, by hand; you should not just do it numerically on a computer. (But you don't have to simplify the final fractions if you don't want to.)

**(b)** [**3**] Compute $f_{21}$. (Hint: perhaps set $C = \{1, 2, 3\}$.)

**2.** Consider a Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities $p_{11} = 1/6$, $p_{12} = 1/3$, $p_{13} = 1/2$, $p_{22} = p_{33} = 1$, and $p_{ij} = 0$ otherwise.

**(a)** [**3**] Compute (with explanation) $f_{12}$.

**(b)** [**2**] Prove that $p_{12}^{(n)} \geq 1/3$, for all positive integers $n$.

**(c)** [**2**] Compute $\sum_{n=1}^{\infty} p_{12}^{(n)}$.

**(d)** [**2**] How do the answers in parts (c) and (a) related to the implication (1) $\Longrightarrow$ (5) in the Stronger Recurrence Theorem?

**3.** Suppose a fair six-sided die is repeatedly rolled, at times $0, 1, 2, 3, \ldots$. (So, each roll is independently equally likely to be 1, 2, 3, 4, 5, or 6.) Let $X_n$ be the largest value that appears among the rolls at times $0, 1, 2, \ldots, n$.

**(a)** [**6**] Find (with justification) a state space $S$, initial probabilities $\{\nu_i\}$, and transition probabilities $\{p_{ij}\}$, with respect to which $\{X_n\}$ is Markov chain.

**(b)** **[3]** Compute the two-step transition probabilities $\{p_{ij}^{(2)}\}$ for all $i, j \in S$.

**(c)** **[4]** Compute the three-step transition probabilities $\{p_{ij}^{(3)}\}$ for all $i, j \in S$.

**4.** For each of the following sets of conditions, either provide (with explanation) an example of Markov chain transition probabilities $\{p_{ij}\}$ on some state space $S$ such that the conditions are satisfied, or prove that no such Markov chain exists.

**(a)** **[3]** $3/4 < p_{12}^{(n)} < 1$ for all $n \geq 1$.

**(b)** **[3]** $p_{11} > 1/2$, and the state 1 is transient.

**(c)** **[3]** $p_{11} > 1/2$, and the state 1 has period 2.

**(d)** **[3]** $f_{12} = 1/2$, and $f_{13} = 2/3$.

**(e)** **[3]** $S$ is finite, and $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$.

**(f)** **[3]** $p_{11} > 1/2$, and the period of state 2 equals 3, and the chain is irreducible.

**(g)** **[3]** $p_{12}^{(n)} \geq 1/4$ and $p_{21}^{(n)} \geq 1/4$ for all $n \geq 1$, and the state 1 is transient.

**5.** Consider a Markov chain with $S = \{1, 2, 3, 4, 5, 6, 7\}$, and transition probabilities

$$(p_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/5 & 4/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/10 & 0 & 0 & 0 & 7/10 & 0 & 1/5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

**(a)** **[7]** Which states are recurrent and which are transient?

**(b)** **[7]** Compute $f_{i1}$ for each $i \in S$. (Hint: leave $f_{41}$ until last.)

**6.** **[5]** Let $S = \mathbf{Z}$ (the set of all integers), and let $h : S \to (0, 1)$ with $h(i) > 0$ for all $i \in S$, and $\sum_{i\in S} h(i) = 1$. Consider the transition probabilities on $S$ given by $p_{ij} = (1/4)\min(1,\ h(j)/h(i))$ if $j = i-2, i-1, i+1$, or $i+2$, and $p_{ii} = 1 - p_{i,i-2} - p_{i,i-1} - p_{i,i+1} - p_{i,i+2}$, and $p_{ij} = 0$ whenever $|j - i| \geq 3$. Prove that $\lim_{n\to\infty} p_{ij}^{(n)} = h(j)$ for all $i, j \in S$. (Carefully justify each step.)

**[END; total points = 70]**