STA 447 / 2006, Winter 2012, Mid-Term Test: SOLUTIONS.

1. Consider a Markov chain with state space $S = \{1, 2\}$, and transition probabilities $p_{11} = 2/3$, $p_{12} = 1/3$, $p_{21} = 1/4$, and $p_{22} = 3/4$.

(a) [3] Compute $p_{12}^{(2)}$.

> **Solution.** $p_{12}^{(2)} = \sum_{k \in S} p_{1k} p_{k2} = p_{11}p_{12} + p_{12}p_{22} = (2/3)(1/3) + (1/3)(3/4) = 2/9 + 1/4 = 17/36$.

(b) [2] Determine whether or not this chain is irreducible.

> **Solution.** *Yes, it's irreducible: since $p_{ij} > 0$ for all $i, j \in S$, therefore $i \to j$ for all $i, j \in S$.*

(c) [4] Let $\pi_1 = 3/7$ and $\pi_2 = 4/7$. Prove that $\{\pi_i\}$ is a stationary probability distribution for this chain.

> **Solution.** *(i) Check that $\pi_i \geq 0$ (clear). (ii) Check that $\pi_1 + \pi_2 = 1$: $(3/7) + (4/7) = 7/7 = 1$. (iii) Check that $\pi_1 p_{11} + \pi_2 p_{21} = \pi_1$: $(3/7)(2/3) + (4/7)(1/4) = 2/7 + 1/7 = 3/7 = \pi_1$. (iv) Check that $\pi_1 p_{12} + \pi_2 p_{22} = \pi_2$: $(3/7)(1/3) + (4/7)(3/4) = 1/7 + 3/7 = 4/7 = \pi_2$. So, $\{\pi_i\}$ is a stationary distribution. (Or, use reversibility.)*

1. (cont'd) Recall that $S = \{1, 2\}$, $p_{11} = 2/3$, $p_{12} = 1/3$, $p_{21} = 1/4$, and $p_{22} = 3/4$.

(d) [3] Determine whether or not $f_{11} = 1$.

> **Solution.** *Yes. Since the chain is irreducible, and has a stationary distribution, it is recurrent (by the Stationary Recurrence Lemma). Hence, $f_{ii} = 1$ for all $i \in S$. In particular, $f_{11} = 1$. (Or, use the Finite Space Theorem. Or compute it directly.)*

(e) [3] Determine whether or not $f_{21} = 1$.

> **Solution.** *Yes. Since $f_{11} = 1$ and $1 \to 2$ (by irreducibility), it follows from the F-Lemma (with $i = 2$ and $j = 1$) that $f_{21} = 1$. (Or compute it directly.)*

(f) [3] Determine whether or not $\sum_{n=1}^{\infty} p_{21}^{(n)} = \infty$.

> **Solution.** *Yes it is. By the Stronger Recurrence Theorem, since the chain is irreducible and $f_{11} = 1$, therefore $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$ for all $i, j \in S$, and in particular $\sum_{n=1}^{\infty} p_{21}^{(n)} = \infty$.*

(g) [3] Determine whether or not this chain is aperiodic.

> **Solution.** *Yes, it's aperiodic: since $p_{ii} > 0$ for all $i \in S$, therefore every state i has period 1.*

(h) [2] Determine whether or not $\lim_{n \to \infty} p_{12}^{(n)} = \pi_2$.

> **Solution.** *Yes it does. The chain is irreducible, aperiodic, and has a stationary distribution $\{\pi_i\}$, so by the Markov Chain Convergence Theorem, $\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j$ for all $i, j \in S$. Setting $i = 1$ and $j = 2$ gives the result.*

2. Consider a Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities $p_{11} = p_{12} = p_{22} = p_{23} = p_{32} = p_{33} = 1/2$, with $p_{ij} = 0$ otherwise.

(a) [3] Determine whether or not this chain is irreducible.

Solution. *No, it isn't: since* $p_{21} = 0$ *and* $p_{31} = 0$, *it is impossible to ever get from state 2 to state 1, so the chain is* not *irreducible.*

(b) [4] Compute f_{ii} for each $i \in S$.

Solution. $f_{11} = p_{11} = 1/2$, *since once we leave state 1 then we can never return to it. But* $f_{22} = 1$, *since from state 2 we will either return to state 2 immediately, or go to state 3 but from there eventually return to state 2. Similarly,* $f_{33} = 1$, *since from state 3 we will either return to state 3 immediately, or go to state 2 but from there eventually return to state 3.*

(c) [3] Specify which states are recurrent, and which states are transient.

Solution. *Since* $f_{11} = 1/2 < 1$, *therefore state 1 is transient. But since* $f_{22} = f_{33} = 1$, *therefore states 2 and 3 are recurrent.*

(d) [3] Compute the value of f_{13}.

Solution. *From state 1, the chain might stay at state 1 for some number of steps, but with probability 1 will eventually move to state 2. Then, from state 2, the chain might stay at state 2 for some number of steps, but with probability 1 will eventually move to state 3. So,* $f_{13} = 1$.