STA447/2006 (Stochastic Processes), Winter 2012

Homework #1

Due: In class by 6:10 p.m. sharp on Thursday February 9. **Warning:** Late homeworks, even by one minute, will be penalised (as discussed on the course web page).

Note: You are welcome to discuss these problems in general terms with your classmates. However, you should figure out the details of your solutions, and write up your solutions, entirely on your own. Directly copying other solutions is strictly prohibited!

[Point values are indicated in square brackets. It is very important to **EXPLAIN** all your solutions very clearly – correct answers poorly explained will **NOT** receive full marks.]

Include at the top of the first page: Your name and student number, and whether you are enrolled in STA447 or STA2006.

1. [5] Consider a (discrete-time) Markov chain $\{X_n\}$ on the state space $S = \{1, 2, 3, 4\}$, with transition probabilities given by

$$(p_{ij}) = \begin{pmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

Suppose $\mathbf{P}(X_0 = 4) = 1$. Compute $\mathbf{P}(X_2 = 3)$. (Explain your reasoning.)

2. Consider a Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities $p_{11} = 1/6$, $p_{12} = 1/3$, $p_{13} = 1/2$, $p_{22} = p_{33} = 1$, and $p_{ij} = 0$ otherwise.

(a) [4] Compute (with explanation) f_{12}.

(b) [2] Prove that $p_{12}^{(n)} \geq 1/3$, for any positive integer n.

(c) [2] Compute $\sum_{n=1}^{\infty} p_{12}^{(n)}$.

(d) [2] Relate the answers in parts (a) and (c) to theorems from class about when $f_{ij} = 1$ and when $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$.

3. Suppose a fair six-sided die is repeatedly rolled, at times $0, 1, 2, 3, \ldots$. (So, each roll is independently equally likely to be 1, 2, 3, 4, 5, or 6.) Let X_n be the largest value that appears among the rolls at times $0, 1, 2, \ldots, n$.

(a) [10] Show that $\{X_n\}$ is a Markov chain, and specify S, $\{\nu_i\}$, and $\{p_{ij}\}$.

(b) [5] Compute the two-step transition probabilities $\{p_{ij}^{(2)}\}$.

4. For each of the following sets of conditions, either provide (with explanation) an example of Markov chain transition probabilities $\{p_{ij}\}$ on some state space S such that the conditions are satisfied, or prove that no such Markov chain exists.

(a) [5] $p_{11} > 1/2$, and the state 1 is transient.

(b) [5] $p_{11} > 1/2$, and the state 1 has period 2.

(c) [5] $p_{11} > 1/2$, and the state 2 has period 2.

(d) [5] $p_{12} = 0$ and $p_{12}^{(3)} = 0$, but $0 < p_{12}^{(2)} < 1$.

(e) [5] $f_{12} = 1/2$, and $f_{13} = 2/3$.

(f) [5] S is finite, and $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in S$.

5. Consider a Markov chain with $S = \{1, 2, 3, 4, 5, 6, 7\}$, and transition probabilities

$$(p_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/5 & 4/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/10 & 0 & 0 & 0 & 7/10 & 0 & 1/5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) [10] Which states are recurrent and which are transient?

(b) [10] Compute f_{i1} for each $i \in S$. (Hint: leave f_{41} until last.)