STA3431H (Monte Carlo Methods), Winter 2011

Homework #1

Due: In class by 2:10 p.m. **sharp** on Monday February 14.

GENERAL NOTES:

- **Late homeworks, even by one minute, will be penalised!**
- Include at the top of the first page: Your name and student number and department and program and year.
- Homework assignments are to be solved by each student individually. You may discuss assignments in general terms with other students, but you must solve it on your own, including doing all of your own computing and writing.
- For full points you should provide very complete solutions, including explaining all of your reasoning clearly and neatly, performing detailed Monte Carlo investigations including multiple runs as appropriate, justifying all of the choices you make, etc.
- You may use results from lecture, but clearly state when you are doing so.
- When writing computer programs for homework assignments:
  - R is the "default" computer programming language and should normally be used for homework; under some circumstances it may be permitted to use other standard computer languages, but only with prior permission from the instructor.
  - You should include both the complete source code and the program output.
  - Programs should be clearly explained, with comments, so they are easy to follow.

THE ACTUAL ASSIGNMENT:

1. Consider the Buffon needle problem, but now with $w < \ell < 2w$. Let Y be the number of parallel lines that the needle lands touching. (So, now it is possible that $Y = 0$ or 1 or 2.) Compute $\mathbf{E}(Y)$. [Hint: regard the needle as the union of smaller needles.]

2. For each of the following choices of parameters, determine whether or not the corresponding Linear Congruential Generator has full period (m).

(a) $m = 24$, $a = 12$, $b = 5$.

(b) $m = 24$, $a = 9$, $b = 5$.

(c) $m = 24$, $a = 13$, $b = 5$.

(d) $m = 24$, $a = 13$, $b = 6$.

3. Let U_1 and U_2 be two independent Uniform$[0, 1]$ random variables, and let $X = U_1/U_2$ and $Y = U_1 + U_2$. Compute the joint probability density function $f_{X,Y}(x, y)$. (Be sure to specify the range of pairs (x, y) for which the density is non-zero.)

4. Write and run a computer program to compute a Monte Carlo estimate (including standard error) of $\mathbf{E}(Y/(1+|Z|))$, where $Y \sim$ Exponential(3) and $Z \sim$ Normal(0, 1) are independent, <u>without</u> using any built-in functions for random number generation or statistical computation (e.g. runif, rnorm, mean, var, sd, etc.). That is, you should just use simple computer commands like variable assignment, arithmetic, log/exp/sin/cos/for/if/while/etc., but you should write your <u>own</u> uniform pseudorandom number generator (of your choice) and your <u>own</u> normal-distribution transformation and exponential-distribution transformation, and your <u>own</u> Monte Carlo routine, and your <u>own</u> computation of mean/variance.

5. Re-write the integral

$$I := \int_1^\infty \left(\int_{-\infty}^\infty (1 + x^2 + \sin(x))^{-|y|^3 - 2}\, dy \right) dx$$

as some expected value, and then estimate I using a Monte Carlo algorithm.

For the remaining questions, let A, B, C, and D be the last four digits of your student number. (So, for example, if your student number were 840245070*, then $A = 5$, $B = 0$, $C = 7$, and $D = 0$.) And, let $g : \mathbf{R}^5 \to [0, \infty)$ be the function defined by:

$$g(x_1, x_2, x_3, x_4, x_5) \;=\; (x_1 + A + 2)^{x_2 + 3} \Big(1 + \cos[(B+3)x_3]\Big) (e^{(12-C)x_4}) |x_4 - 3x_5|^{D+2} \prod_{i=1}^5 \mathbf{1}_{0 < x_i < 2}\,,$$

and let $\pi(x_1, x_2, x_3, x_4, x_5) = c\, g(x_1, x_2, x_3, x_4, x_5)$ be the corresponding five-dimensional probability density function, with unknown normalising constant c. Finally, let f be the uniform density on $[0, 2]^5$.

6. Identify the values of A, B, C, and D. (This should be easy!)

7. Write and run a program to estimate $\mathbf{E}_\pi[(X_1 - X_2)/(1 + X_3 + X_4 X_5)]$ by using an importance sampler with the above f. Either obtain an accurate estimate this way, or explain why this computation is too difficult to succeed.

8. Write and run a program to estimate $\mathbf{E}_\pi[(X_1 - X_2)/(1 + X_3 + X_4 X_5)]$ by using a rejection sampler with the above f. Either obtain an accurate estimate this way, or explain why this computation is too difficult to succeed.

9. Write and run a program to estimate $\mathbf{E}_\pi[(X_1 - X_2)/(1 + X_3 + X_4 X_5)]$ using an MCMC algorithm of your choice, and obtain the best estimate you can. Include some discussion of accuracy, uncertainty, standard errors, etc.

---

* (Historical note: this was the instructor's actual student number when he was a UofT undergraduate student in 1984–88.)