CHAPTER ONE. Introduction.

My Ph.D. thesis is concerned with convergence properties of certain Markov chains. Specifically, I have studied various random walks on finite and compact Lie groups (Chapters 2 and 3), and have also studied certain Markov chains arising from stochastic algorithms used in Bayesian Statistics (Chapters 4 and 5).

Each of the Markov chains studied has a stationary distribution to which it converges. For a random walk on a finite group, this stationary distribution is the uniform distribution on the group. For a compact Lie group, this stationary distribution is normalized Haar measure. For the stochastic algorithms used in Statistics, the stationary distribution is more complicated and is, in fact, the posterior distribution that the algorithm aims to approximate. In each case, the main question that we study is how quickly the distribution of the Markov chain approaches this stationary distribution. This is a very natural question to ask, and is particularly appropriate for the stochastic algorithms, because these algorithms are actually being run on computers and a major source of confusion concerns when to tell the computer to stop.

The total variation (L^1) norm is used to measure the convergence to stationarity. For each Markov chain studied, we determine how many iterations of the Markov chain are required until the total variation distance to the given stationary distribution is appropriately small. Each rate of convergence is obtained in terms of auxilary parameters related to the size and/or the complexity of the Markov chain under consideration. A particular emphasis is how this rate of convergence grows asymptotically as these parameters get large.

The methods used to analyze the convergence depend to some extent on the Markov chain under consideration. For the random walks on groups, we use Fourier analysis almost exclusively. Specifically, the Upper Bound Lemma of Diaconis and Shashahani allows one to get upper bounds on the distance to stationarity in terms of the Fourier transforms of the step distribution, evaluated at the irreducible representations of the group. We apply the Upper Bound Lemma to certain specific random walks on the orthogonal group (Chapter 2) and on the circle groups (Chapter 3), to get sharp upper bounds on the time to stationarity in terms of the size of the group.

Irreducible characters of the group can also combine with Chebychev's Inequality to produce lower bounds on the time to stationarity (as discussed in Chapter 2). If the upper and lower bounds coincide to first order in the size of the group, then we have a "Cut-off Phenomenon" in the terminology of Aldous and Diaconis. In Chapter 2, working on the orthogonal group, we obtain the first rigorous example of a Cut-off Phenomenon on a non-finite group.

When the Markov chain under consideration is not a random walk on a group, Fourier analysis is not available. Instead, we use ideas related to Doeblin- and Harris-recurrence (and, ultimately, to the Coupling Inequality) to develop new and very general methods of obtaining upper bounds on the distance to stationarity. These methods, combined with a careful analysis of the specific Markov chain being studied, lead to useful information about the distance to stationarity. We apply these ideas to obtain fairly sharp rates of convergence for the Data Augmentation (Chapter 4) and Gibbs Sampler (Chapter 5) Markov chains that are being used in calculations of posterior distributions in Bayesian Statistics. Our analysis is for fairly specific situations, but the methods used appear to be quite widely applicable.

Finally, in Chapter 6 we offer some concluding remarks and some directions for further research.

The mathematical background required to read this thesis is only moderately involved. Some basic familiarity with probabilistic notation and with measure theory are necessary throughout. In Chapters 2 and 3, some previous knowledge of representation theory would be an asset. In Chapters 4 and 5, it would be helpful (but not absolutely required) to have some acquaintance with Bayesian Statistics and with Harris Recurrence and Coupling. In any case, we have tried to make the presentation clear and self- contained, and to separate out the many detailed calculations from the essential ideas involved. Each chapter contains its own reference section with material for further reading. Enjoy!