Summary of Readings on Random Walks on Graphs

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This is a summary of readings from the paper Random Walks on Graphs: A Survey by L. Lovász. Readings were done under the guidance of Professor J. Rosenthal whose course STA447 Stochastic Processes serves as the main source of background knowledge and motivation.

1 Definitions and Notation

Let $G = (V, E)$ be an unweighted graph, where V denotes the set of vertices and E denotes the set of edges. The **degree of a vertex** is the number of edges incident to the vertex. Denote the degree of a vertex $u \in V$ by $d(u)$. A graph is **regular** if every vertex has the same degree. A graph is **complete** if there is an edge between every pair of vertices.

Define a Markov chain with state space V and transition probabilities

$$p_{uv} = \begin{cases} \frac{1}{d(u)}, & \text{if } uv \in E \\ 0, & \text{otherwise} \end{cases}$$

As shown in class, the stationary distribution is

$$\pi(u) = \frac{d(u)}{Z}$$

where $Z = \sum_{u \in V} d(u)$. For unweighted graphs, $Z = 2 \times$ number of edges.

The **access time** from vertex u to v is the expected number of steps to reach v, starting at u. Denote the access time by $H(u, v)$. The **commute time** is $\kappa(u, v) = H(u, v) + H(v, u)$. The **cover time** is the expected number of steps to reach every vertex in the graph.

Remark: "vertex" and "node" are used interchangeably.

2 Time Reversibility

If G is regular, then the graph is symmetric, and so is the Markov chain. The probability of moving from node u to v is the same as the probability of moving from v to u. If G is not regular, then we say it is time reversible.

Time reversibility is the notion that the long run probability of a Markov chain going from state u to v is the same as the long run probability of going from state v to u. Given a Markov chain

$\{X_n\}$ with state space S and stationary distribution π, when in stationarity, we have $X_n \sim \pi$ for all n. Then

$$P(X_n = u, X_{n+1} = v) = P(X_{n+1} = v | X_n = u) P(X_n = u) = p_{uv}\pi_u$$
$$P(X_n = v, X_{n+1} = u) = P(X_{n+1} = u | X_n = v) P(X_n = v) = p_{vu}\pi_v$$

And so the Markov chain is time reversible w.r.t. to π if $p_{uv}\pi_u = p_{vu}\pi_v$. For an alternate approach to time reversibility, see Appendix A.

So far we have talked about stationary distribution for vertices. We can also extend this to edges. Define a new Markov chain $Z_n = (X_n, X_{n+1})$ with state space $S \times S$. Consider the edge $uv \in E$, and $P(Z_n = (u, v))$. This is the probability that the Markov chain passes through the edge uv, which is the probability that we first get to node u, then from u to v. So in fact

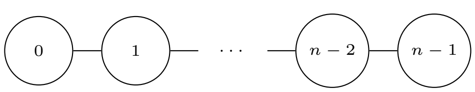
$$P(Z_n = (u, v)) = \pi_u p_{uv} = \frac{d(u)}{Z}\frac{1}{d(u)} = \frac{1}{Z}$$

The probability that this random walk passes any edge is $\frac{1}{Z}$, so on average, we expect the random walk to return to an edge every Z steps.

3 Access Times and Cover Times

3.1 Paths

Suppose we have a path with n nodes, from 0 to $n-1$.



Claim 1. $H(k-1, k) = 2k - 1$.
Proof. Consider a path with $k+1$ nodes, from 0 to k. Suppose we start at the last node k. Let $T = \inf\{m \geq 1 : X_m = k\}$, so $E_k(T)$ is the expected return time of this random walk. Then

1. $E_k(T) = 2k$
   From class, we showed that the stationary distribution of this graph is $\pi_i = \frac{d(i)}{Z}$, where $Z = 2k$. Since $d(k) = 1$, we have $\pi_k = \frac{1}{Z}$. By the Recurrence Time Theorem, it takes on average $\frac{1}{\pi_k} = Z = 2k$ steps to return to k [5, p. 31].

2. $H(k-1, k) = E_k(T) - 1$
   Since we start at the last node k, which has degree 1, the only possible node to go to is node $k-1$. Once we are at node $k-1$, the expected time to return to k is precisely $H(k-1, k)$. So the expected return time $E_k(T)$, starting at k, is $1 + H(k-1, k)$.

Combining the above two, we get that $H(k-1, k) = E_k(T) - 1 = 2k - 1$.

Claim 2. $H(i, k) = k^2 - i^2$ for $0 \leq i < k \leq n$.
Proof. Let $0 \leq i < k \leq n$, and consider the access time $H(i, k)$. Starting at node i, to reach node k, we have to first get to node $k-1$, which takes around $H(i, k-1)$ steps. From $k-1$ to k takes $H(k-1, k) = 2k - 1$ steps, by above claim. So

$$H(i, k) = H(i, k-1) + H(k-1, k) = H(i, k-1) + 2k - 1$$

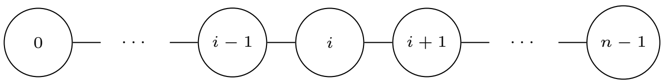
Solving for this recurrence relation we get

$$H(i,k) = k^2 - i^2$$

See detailed proof in Appendix B.

Claim 3. Starting from node 0, the cover time of a path of length n is $(n-1)^2$.
Proof. In this case, the cover time is $H(0, n-1)$, which is $(n-1)^2$ by Claim 2.

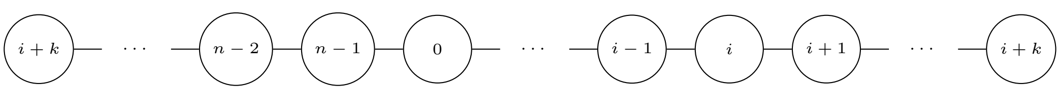
Claim 4. Starting from any internal node i, the cover time of a path of length n is $i(n-1-i) + (n-1)^2$.
Proof. Suppose we start at an internal node i.



The cover time is the time to first get to one end of the path (either node 0 or node $n-1$) and then reach the other end. Getting from one end to the other is simply $H(0, n-1) = (n-1)^2$ from Claim 3. To first get to one end, let $T = \inf\{m \geq 0 : X_m = 0 \text{ or } n-1\}$. We can view this as a version of Gambler's Ruin, where we start with $a = i$ dollars, and the game finishes when we get to either 0 or $c = n-1$ dollars. Since $p = \frac{1}{2}$, we showed in class that $E(T) = a(c-a) = i(n-1-i)$ [5, p. 38]. So the cover time is $i(n-1-i) + (n-1)^2$.

3.2 Circuits

Claim 1. The access time between two nodes of distance k of a circuit of length n is $k(n-k)$.
Proof. Suppose we have a circuit of length n, with nodes 0 to $n-1$. Consider two nodes i and $i+k$ on the circuit. The figure below represents the circuit in a straight line with the two ends denoting the same node $i+k$.



Note that if we start at i, then we can reach $i+k$ either by going k steps to the right or $n-k$ steps to the left. Let $T = \inf\{m \geq 0 : X_m = i+k\}$. Again, we can use Gambler's Ruin with $p = \frac{1}{2}$ to model this problem. We relabel the above $n+1$ nodes from left to right as $0, 1, ..., n$. Then $T = \inf\{m \geq 0 : X_m = 0 \text{ or } n\}$. Using notation from class, we have $a = n-k$ and $c = n$. Then by corollary in lecture notes, $E(T) = a(c-a) = (n-k)k$ [5, p. 38].

Claim 2. The cover time of a circuit of length n is $\frac{n(n-1)}{2}$.
Proof. Let $f(n)$ denote the cover time of a circuit with n nodes. To reach all n nodes, we must first reach $n-1$ nodes, which takes on average $f(n-1)$ steps. Once we have visited the $(n-1)$st node, there is one node remaining. Note that the $(n-1)$st node and the last node are adjacent to each other. This is because at every step, we can only go to the left or to the right of the current node, so the set of nodes already visited always forms a connected path. Now, since the $(n-1)$st node and the last node are one node apart, the access time between these two nodes is $n-1$, by Claim 1. So the cover time is

$$f(n) = f(n-1) + (n-1)$$

Solving this recurrence relation gives

$$f(n) = \frac{n(n-1)}{2}$$

See detailed proof in Appendix C.

3.3 Complete Graphs

Claim 1. Consider a complete graph with n nodes from 0 to $n-1$. Then $H(i,j) = n-1$ for all nodes $i, j \in \{0, ..., n-1\},\ i \neq j$.

Proof. Note that a complete graph is symmetrical, and so to compute the access time between any two nodes, we can simply compute $H(0,1)$. So suppose we start at node 0. To reach node 1 for the first time on the t-th step requires the first $t-1$ steps to be any node other than node 1. Hence, at any stage of the first $t-1$ steps, we always have $n-2$ choices (n nodes minus node 1 and the current node). Now let T be the first time we reach node 1. The probability of doing so on the t-th step is

$$P(T = t) = \left(\frac{n-2}{n-1}\right)^{t-1} \frac{1}{n-1}$$

So the expected value of T is

$$E(T) = H(0,1) = \sum_{t=1}^{\infty} tP(T = t) = \sum_{t=1}^{\infty} t \left(\frac{n-2}{n-1}\right)^{t-1} \frac{1}{n-1} = n-1$$

See detailed proof of last equality in Appendix D.

Claim 2. The cover time for a complete graph with n nodes is approximately $n \log n$.

Proof. Let τ_i denote the first time we hit i vertices. So $\tau_1 = 0, \tau_2 = 1$, and $\tau_1 < \tau_2 < \cdots < \tau_n$. We want to find $E(\tau_n)$. Note that $\tau_{i+1} - \tau_i$ is the number of steps until we hit a new vertex. The probability of hitting a new vertex is $\frac{n-i}{n-1}$ since we have $n-1$ choices, and i nodes have already been visited. So on average, it takes $\frac{n-1}{n-i}$ steps to reach a new vertex i.e.

$$E(\tau_{i+1} - \tau_i) = \frac{n-1}{n-i}$$

See detailed proof in Appendix E. Then

$$\begin{aligned} E(\tau_n) &= E(\tau_n - \tau_{n-1} + \tau_{n-1} - \cdots - \tau_1) \\ &= \sum_{i=1}^{n-1} E(\tau_{i+1} - \tau_i) = \sum_{i=1}^{n-1} \frac{n-1}{n-i} \approx n \log n \end{aligned}$$

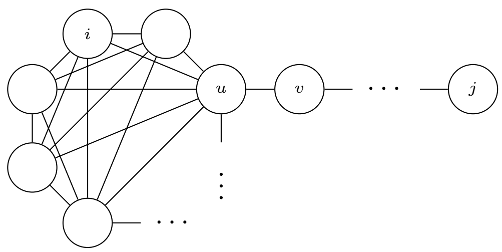
See detailed proof of last approximation in Appendix E.

3.4 Other Graphs

3.4.1 Lollipop Graphs

Consider a graph made up of a complete subgraph of $\frac{n}{2}$ vertices attached to a path of length $\frac{n}{2}$. Let u be the node connecting the complete subgraph and the path. Let i be a node, other than

u, of the complete subgraph. Let v be the node adjacent to u on the path, and let j be the other endpoint of the path.



Claim 1. $H(i, j) = \Omega(n^3)$.
Proof. By Section 3.3 Claim 1, since the complete subgraph has $\frac{n}{2}$ nodes, $H(i, u) = \frac{n}{2} - 1$. Node u has degree $1 + (\frac{n}{2} - 1) = \frac{n}{2}$, so we move back to a node of the complete subgraph with probability $\frac{n/2-1}{n/2} = 1 - \frac{2}{n}$, and move to v with probability $\frac{2}{n}$. So to get from u to v, we either get there in one step, with probability $\frac{2}{n}$, or we move back to some node $k \neq u$ of the complete subgraph, in which case we need to come back to u and then to v. Hence,

$$H(u, v) = \frac{2}{n} + \left(1 - \frac{2}{n}\right)(H(k, u) + H(u, v))$$
$$\Rightarrow \left(\frac{2}{n}\right) H(u, v) = \frac{2}{n} + \left(1 - \frac{2}{n}\right) H(k, u) = \frac{2}{n} + \left(\frac{n-2}{n}\right)\left(\frac{n}{2} - 1\right)$$
$$\Rightarrow H(u, v) = 1 + \frac{n}{2}\left(\frac{n-2}{n}\right)\left(\frac{n}{2} - 1\right) = 1 + \left(\frac{n-2}{2}\right)^2 = \Omega(n^2)$$

Now, $H(i, v) = H(i, u) + H(u, v)$ since the only way to get from i to v is to reach u first, then go to v. So $H(i, v) = \left(\frac{n}{2} - 1\right) + \left(1 + \left(\frac{n-2}{2}\right)^2\right) = \Omega(n^2)$.

Since the path has $\frac{n}{2}$ nodes, rename the rest of the nodes starting at v: $v, v+1, v+2, ..., v+\frac{n}{2}-1 = j$. Then $p_{v,v+1} = p_{v,u} = \frac{1}{2}$. To get from v to $v + 1$, we either do so in one step, with probability $\frac{1}{2}$, or we move back to u, then from u to v to $v + 1$. So

$$H(v, v + 1) = \frac{1}{2} + \frac{1}{2}(H(u, v) + H(v, v + 1))$$
$$\Rightarrow H(v, v + 1) = 1 + H(u, v)$$

Similarly,

$$H(v + 1, v + 2) = \frac{1}{2} + \frac{1}{2}(H(v, v + 1) + H(v + 1, v + 2))$$
$$\Rightarrow H(v + 1, v + 2) = 1 + H(v, v + 1) = 1 + 1 + H(u, v) = 2 + H(u, v)$$

Continuing recursively for the rest of the path, we get that

$$H(v + i, v + i + 1) = i + 1 + H(u, v)$$

for $0 \leq i \leq \frac{n}{2} - 2$.

Note that $H(v,j) = H(v, v+\frac{n}{2}-1) = H(v,v+1) + H(v+1,v+2) + \cdots + H(v+\frac{n}{2}-2, v+\frac{n}{2}-1)$. So

$$\begin{aligned}
H(v,j) &= \sum_{i=0}^{\frac{n}{2}-2} H(v+i, v+i+1) = \sum_{i=0}^{\frac{n}{2}-2} i+1+H(u,v) = \sum_{i=0}^{\frac{n}{2}-2} i + \sum_{i=0}^{\frac{n}{2}-2} 1 + \sum_{i=0}^{\frac{n}{2}-2} H(u,v) \\
&= \frac{(\frac{n}{2}-2)(\frac{n}{2}-1)}{2} + \frac{n}{2} - 1 + \left(\frac{n}{2}-1\right) H(u,v) \\
&= \frac{1}{2}\left(\frac{n-4}{2}\right)\left(\frac{n-2}{2}\right) + \frac{n}{2} - 1 + \left(\frac{n}{2}-1\right)\left(1 + \left(\frac{n-2}{2}\right)^2\right) = \Omega(n^3)
\end{aligned}$$

Finally, we have $H(i,j) = H(i,v) + H(v,j) = \Omega(n^3)$.

Remark: Another way of approaching the question is to define a new Markov chain, with state space $S = \{1,2,3,4\}$ where 1 is the set of nodes (excluding u) of the complete subgraph, 2 is the node u, 3 is the node v and 4 is the node j. Then we have transition probabilities

$$\begin{bmatrix}
(n-4)/(n-2) & 2/(n-2) & 0 & 0 \\
(n-2)/n & 0 & 2/n & 0 \\
0 & (n-2)/n & 0 & 2/n \\
0 & 0 & 2/(n-2) & (n-4)/(n-2)
\end{bmatrix}$$

See detailed computation of the transition probabilities in Appendix F.

From the matrix, we see that the expected time from each state to the next is $\Omega(n)$ i.e. let $E_a(T_b)$ be the expected time to get from a to b. Then $E_1(T_2) = E_2(T_3) = E_3(T_4) = \Omega(n)$ and so $E_1(T_4) = \Omega(n^3)$. Here we multiply instead of add the $\Omega(n)$'s since at each step, there is the possibility of returning to state 1, in which case it would take another $\Omega(n)$ steps to reach the current state.

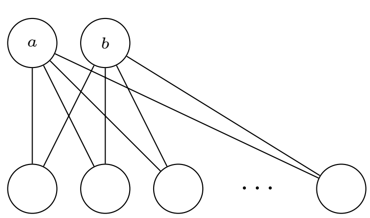
Alternatively, since we are done once we reach state 4, the transition probabilities of state 4 are irrelevant. We could let $p_{41} = 1$ and $p_{4i} = 0$ for $i \neq 1$. Then $E_4(T_4) = 1 + E_1(T_4)$, so $E_1(T_4) = E_4(T_4) - 1$. But $E_4(T_4)$ is just the mean recurrence time of state 4, which is $\frac{1}{\pi_4}$. Hence we get

$$E_1(T_4) = \frac{1}{\pi_4} - 1 = \frac{1}{8/(n^3 - 2n^2 + 12n)} - 1 = \Omega(n^3)$$

See detailed computation of π_4 in Appendix F.

3.4.2 Complete Bipartite Graphs

Consider a complete bipartite graph $K_{2,n}$, with 2 nodes on one side, and n nodes on the other.



Claim 1. Let a, b be the two nodes on the side with only two nodes, as depicted in the above figure. Then $\kappa(a, b) = 8$.
Proof. Recall that $\kappa(a, b)$ is the commute time, defined by $\kappa(a, b) = H(a, b) + H(b, a)$. Suppose we start from node a. Since the graph is bipartite, we can only return to the same side in an even number of steps. Let $U = \{a, b\}$ and V denote the set of vertices on the other side so $|V| = n$. Note that $p_{u,v} = \frac{1}{n}$ and $p_{v,u} = \frac{1}{2}$ for any $u \in U, v \in V$. Let T_b denote the first time we reach b. Then for any $c \in \mathbb{N}$,

$$\begin{aligned} P_a(T_b = 2c) &= \sum_{v_c \in V} \cdots \sum_{v_1 \in V} p_{a,v_1} p_{v_1,a} \cdots p_{a,v_c} p_{v_c,b} \\ &= \sum_{v_c \in V} \cdots \sum_{v_1 \in V} \frac{1}{n} \times \frac{1}{2} \times \cdots \times \frac{1}{n} \times \frac{1}{2} = \sum_{v_c \in V} \cdots \sum_{v_1 \in V} \left(\frac{1}{2n} \right)^c = \frac{1}{2^c} \\ H(a, b) = E_a(T_b) &= \sum_{c=1}^{\infty} (2c) \frac{1}{2^c} = 4 \end{aligned}$$

Similarly, $H(b, a) = 4$. So $\kappa(a, b) = 8$.

See detailed proof of $H(a, b) = 4$ in Appendix G.

3.5 Bounds on Commute Time

Let G be a graph, let i, j be any two nodes of the graph, and let m be the number of edges in G.

Claim 1. If i, j are adjacent, then $\kappa(i, j) \leq 2m$.
Proof. We have $\kappa(i, j) = H(i, j) + H(j, i)$. Since i, j are adjacent, ji is an edge of the graph. Recall from Section 2 that if we are on an edge ji, then the expected time before traversing it again in the same direction is $Z = 2m$. But traversing ji is only one of the possibly many ways to reach i from j. Hence $\kappa(i, j) \leq 2m$.

Claim 2. If i, j are at distance r, then $\kappa(i, j) \leq 2mr < n^3$.
Proof. Let i, j be two nodes at distance r, and let $a_1, ..., a_{r-1}$ be the nodes in between i and j s.t. $ia_1, a_1a_2, ..., a_{r-2}a_{r-1}, a_{r-1}j$ are edges in the graph. We have $\kappa(i, j) = H(i, j) + H(j, i)$, where

$$\begin{aligned} H(i, j) &\leq H(i, a_1) + H(a_1, j) \\ &\leq H(i, a_1) + H(a_1, a_2) + H(a_2, j) \\ &\vdots \\ &\leq H(i, a_1) + H(a_1, a_2) + \cdots + H(a_{r-2}, a_{r-1}) + H(a_{r-1}, j) \end{aligned}$$

The first inequality holds because getting from i to a_1 then a_1 to j is one of possibly many ways to get from i to j. The remaining inequalities follow. Similarly

$$H(j, i) \leq H(j, a_{r-1}) + H(a_{r-1}, a_{r-2}) + \cdots + H(a_2, a_1) + H(a_1, j)$$

Combining the above inequality, we get

$$\begin{aligned}
\kappa(i,j) &= H(i,j) + H(j,i) \\
&\leq (H(i,a_1) + \cdots + H(a_{r-1},j)) + (H(j,a_{r-1}) + \cdots + H(a_1,j)) \\
&\leq (H(i,a_1) + H(a_1,i)) + \cdots + (H(a_{r-1},j) + H(j,a_{r-1})) \\
&\leq \kappa(i,a_1) + \cdots \kappa(a_{r-1},j) \\
&\leq 2mr
\end{aligned}$$

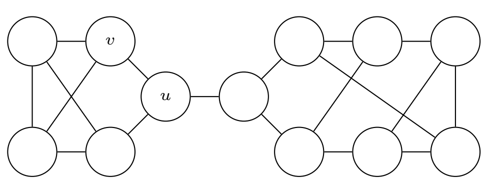
The last inequality uses Claim 1.

Now we show $2mr < n^3$. Clearly if the graph has n nodes, and i and j are at distance r, then $r < n$. Note that for a complete graph with n nodes, there are $\frac{n(n-1)}{2}$ edges. So for an arbitrary graph, the number of edges at most $\frac{n(n-1)}{2}$. So $2m \leq n(n-1) < n^2$, and hence $2mr < n^3$.

3.6 Symmetry and Access Time

In this section, we show that while the access time from i to j may not be equal to the access time from j to i, even for regular graphs, there are still other symmetry properties.

Claim 1. Let G be a regular graph, and let i, j be two nodes of the graph. Then it is not always the case that $H(i,j) = H(j,i)$.
Proof. We provide a counterexample. Consider the following 3-regular graph G with a cutnode u. A cutnode is a node such that when removed from the graph results in more components (e.g. if the graph is originally connected, then removing a cutnode will disconnect the graph).



Let G_1 be the subgraph on the left, and G_2 be the subgraph on the right, including node u. Then $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{u\}$. Note that the access time from v to u in G is the same as that in G_1. If we fix the number of nodes in G_1, then the access time from v to u is bounded. However, we can extend G_2 to be arbitrarily large, so $H(u,v)$ increases independently of $H(v,u)$.

Claim 2. The probability of starting at node u and visiting v before returning to u is $1/\kappa(u,v)\pi(u)$.
Proof. Recall that $\kappa(u,v) = H(u,v) + H(v,u)$. Consider a random walk starting at node u. Let τ be the first time returning to u, and let σ be the first time returning to u after hitting v. Since $\pi(u) = \frac{d(u)}{Z}$, we have previously shown that $E(\tau) = \frac{Z}{d(u)}$. By definition of σ, $E(\sigma) = \kappa(u,v)$.

Note that $\tau \leq \sigma$ since we return to u either without hitting v or after hitting v. The probability in question is $P(\tau = \sigma)$. Let q denote this probability. If the first time we return to u is after hitting v, then $\tau = \sigma$. If not, then after returning to u the first time, we have to start from u again, reach

v, then return to u. So

$$\begin{aligned}E(\sigma) &= qE(\tau) + (1-q)E(\tau+\sigma) \\ &= qE(\tau) + E(\tau+\sigma) - qE(\tau+\sigma) \\ &= E(\tau+\sigma) - qE(\sigma)\end{aligned}$$

Rearrange to get

$$q = \frac{E(\tau)}{E(\sigma)} = \frac{Z}{d(u)\kappa(u,v)} = \frac{1}{\pi(u)\kappa(u,v)}$$

Remark on notation: In class, we let $T_i = \min\{m \geq 1 : X_m = i\}$. So the probability q in this problem is the same as $P_u(T_v < T_u)$. Similarly, we could let A be the event that $T_v < T_u$. Then $P_u(A) = q$ and

$$\sigma - \tau \overset{d}{=} \begin{cases} 0, & \text{on } A \\ \sigma, & \text{on } A^c \end{cases}$$

Then we can compute the expected value to arrive at the same conclusion.

$$\begin{aligned}&E(\sigma-\tau) = q(0) + (1-q)E(\sigma) \\ &\Leftrightarrow E(\tau) = qE(\sigma)\end{aligned}$$

Claim 3. Let u and v be two nodes with the same degree. Then the probability of starting at node u and visiting v before returning to u is equal to the probability of starting at node v and visiting u before returning to v.
Proof. This follows from Claim 1. The probabilities in question are $1/\kappa(u,v)\pi(u)$ and $1/\kappa(v,u)\pi(v)$. Since u and v have the same degree, $\pi(v) = \frac{d(v)}{Z} = \frac{d(u)}{Z} = \pi(u)$. And $\kappa(v,u) = H(v,u) + H(u,v) = H(u,v) + H(v,u) = \kappa(u,v)$. So

$$1/\kappa(u,v)\pi(u) = 1/\kappa(v,u)\pi(v)$$

3.7 Bounds on Cover Time

In this section, we find a bound for the cover time of any graph, w.r.t h, the maximum access time between any two nodes.

Claim 1. Let b be the expected time before more than half of the nodes are visited. Then $b < 2h$.
Proof. Let α_v be the time when node v is visited. Let β be the time when we reach more than half of the nodes. Then β is the $(\lfloor \frac{n}{2} \rfloor + 1)$st largest of the α_v's. After reaching this node, there are $n - \lfloor \frac{n}{2} \rfloor - 1 = \lceil \frac{n}{2} \rceil - 1$ nodes remaining, all of which will have $\alpha_v > \beta$. So

$$\begin{aligned}&\frac{n}{2}\beta \leq \left(\left\lceil \frac{n}{2} \right\rceil - 1 + 1\right)\beta < \sum_v \alpha_v \\ &\Leftrightarrow \beta < \frac{2}{n}\sum_v \alpha_v\end{aligned}$$

Then

$$b = E(\beta) < \frac{2}{n}\sum_v E(\alpha_v) \leq \frac{2}{n}nh = 2h$$

The second inequality is because each $E(\alpha_v) \le h$ by definition of h.

Claim 2. Suppose we have already covered more than $\frac{2^k-1}{2^k}$ of the n nodes. Let b be the expected time before more than half of the remaining nodes are visited. Then $b < 2h$.
Proof. We prove by induction on k and mimic the proof for Claim 1. For $k = 0$, this is essentially Claim 1. For the remainder of the proof, we drop any floor and ceiling functions to avoid messiness.

Now suppose the statement holds for all $k \le l$ where $l \in \mathbb{N}$, and suppose we have covered more than $\frac{2^l-1}{2^l}$ of the nodes. We have at most $n - (\frac{2^l-1}{2^l}n + 1) = \frac{n}{2^l} - 1$ nodes not yet reached. Let U denote the set of these nodes, and for simplicity, suppose $|U| = \frac{n}{2^l} - 1$. Let β be the time when we reach more than half of the nodes in U, which is at least $(\frac{n}{2^l} - 1)\frac{1}{2} + 1 = \frac{n}{2^{l+1}} + \frac{1}{2}$ nodes. Let α_u be the time when node $u \in U$ is visited.

We make the following observation. Once we reach the $\frac{n}{2^{l+1}} + \frac{1}{2}$ nodes by time β, all of these nodes u will have $\alpha_u \le \beta$. As of now, we have not covered every node in the graph. Let w denote the last node reached by time β. There are still $|U| - (\frac{n}{2^{l+1}} + \frac{1}{2}) = \frac{n}{2^l} - 1 - \frac{n}{2^{l+1}} - \frac{1}{2} = \frac{n}{2^{l+1}} - \frac{3}{2}$ nodes in U not yet reached. These remaining $\frac{n}{2^{l+1}} - \frac{3}{2}$ nodes, plus node w, will have $\alpha_u \ge \beta$, since we are already at time β and have not visited them. So

$$\sum_{u \in U} \alpha_u > \left(\frac{n}{2^{l+1}} - \frac{3}{2} + 1\right)\beta = \left(\frac{n - 2^l}{2^{l+1}}\right)\beta$$
$$\Leftrightarrow \beta < \left(\frac{2^{l+1}}{n - 2^l}\right)\sum_u \alpha_u$$

Then

$$b = E(\beta) < \left(\frac{2^{l+1}}{n - 2^l}\right)\sum_u E(\alpha_u) \le \left(\frac{2^{l+1}}{n - 2^l}\right)|U|h = \left(\frac{2^{l+1}}{n - 2^l}\right)\left(\frac{n}{2^l} - 1\right)h$$
$$< \left(\frac{2^{l+1}}{n - 2^l}\right)\left(\frac{n - 2^l}{2^l}\right)h = 2h$$

Claim 3. The cover time from any node of a graph with n nodes is at most $2(\log_2 n)h$.
Proof. From Claim 1, we know that the expected time to cover more than half of the nodes is less than $2h$. From Claim 2, we know that the expected time to cover more than half of the remaining is less than $2h$. And so the expected time to cover more than 3/4 of the nodes is less than $4h$. Continuing inductively, we get that the expected time to reach more than $\frac{2^k-1}{2^k}$ of the nodes is bounded above by $2kh$. In other words, in $2kh$ time, we can cover a graph of size n, where

$$n = n\left(\frac{2^k - 1}{2^k}\right) + 1 = n\left(1 - \frac{1}{2^k}\right) + 1 \Leftrightarrow n = 2^k \Leftrightarrow k = \log_2 n$$

So given a graph of size n where $2^{k-1} < n \le 2^k$, the cover time is less than $2kh = 2(\log_2 n)h$.

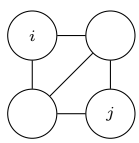
3.8 Monotonicity

One would expect that as we increase the number of edges in a graph, it becomes easier to get from one node to another, and thus access times and commute times should decrease. However, this is not necessarily the case.

Claim 1. Access times and commute times are generally not monotone decreasing w.r.t. the number of edges in a graph.

Proof. We provide some counterexamples. Consider a path with n nodes from 0 to $n-1$. Note that $H(0,1) = 1$ since there is only one way to get from node 0 to node 1. Now suppose we add an edge connecting 0 to $n-1$, hence making the path a circuit. Starting at 0 again, we can go either left or right. If we go to node $n-1$, then it would take at least two steps to reach 1. So $H(0,1) > 1$.

Consider a circuit with 4 nodes. Then since $\pi_i = \frac{2}{8} = \frac{1}{4}$ for all nodes i, the commute time between any two nodes i, j is $\kappa(i,j) = H(i,j) + H(j,i) = 4 + 4 = 8$. Suppose we add an edge connecting two opposite nodes as in the figure below. Now, $\pi_i = \pi_j = \frac{2}{10} = \frac{1}{5}$, and $\kappa(i,j) = 10$. So commute time increased.



Nevertheless, we do have the following "almost monotonicity" property.

Claim 2. Let G be a graph with m edges, and G' be G with an additional edge. Let i, j be two nodes in G'. Then

$$\kappa_{G'}(i,j) \leq \left(\frac{m+1}{m}\right) \kappa_G(i,j)$$

where $\kappa_{G'}(i,j)$ and $\kappa_G(i,j)$ are the commute times of i, j in graphs G', G, respectively.

Proof. The proof of this claim requires us to introduce the idea of a harmonic function, and to view the graph as an electrical network. We show that given two nodes s and t, $\kappa(s,t) = 2mR_{st}$, where R_{st} denotes the resistance between the two nodes. Then we use a result on monotonicity to complete the proof.

Let $G = (V, E)$ be a graph. A function $\phi : V \to \mathbb{R}$ is a **harmonic function with set of poles** $S \subseteq V$ if for all $v \notin S$,

$$\phi(v) = \frac{1}{d(v)} \sum_{u \in \Gamma(v)} \phi(u)$$

where $\Gamma(v)$ denotes the set of neighbours of v. In other words, the function is a weighted average of neighbouring vertices [3]. Consider the following two ways of constructing harmonic functions.

1. Let $\phi(v)$ denote the probability of starting at node v and hitting s before t, where s, t are distinct vertices in V. Then ϕ is harmonic with poles s and t, where $\phi(s) = 1, \phi(t) = 0$.

2. Consider G as an electrical network, where an edge represents a unit resistance. Suppose an electric current is flowing through G, entering at s and leaving at t. Let $\phi(v)$ be the voltage of v. Then ϕ is harmonic with poles s and t, where $\phi(s) = 0, \phi(t) = 1$.

Let R_{st} be the resistance between nodes s and t. Then $\kappa(s,t) = 2mR_{st}$.

Viewing G as an electrical network with a current from s to t as above, we have $\phi_{st}(v)$ as the voltage of v in this graph. The voltage of t is 1, and the total current is given by

$\sum_{u\in\Gamma(t)} \phi_{st}(u)$. Since voltage = current $\times$ resistance by Ohm's law, we have

$$R_{st} = \frac{1}{\sum_{u\in\Gamma(t)} \phi_{st}(u)}$$

At the same time, we can also view $\phi_{st}(v)$ as the probability of a random walking starting at u and hitting s before t. Then $\phi_{st}(t)$ is the probability of starting at t, visiting s before returning to t. Well, $\phi_{st}(t) = \frac{1}{d(t)} \sum_{u\in\Gamma(t)} \phi_{st}(u)$, and by Section 3.6 Claim 2, this probability is equal to $\frac{1}{\kappa(t,s)\pi(t)} = \frac{2m}{\kappa(s,t)d(t)}$. So

$$\begin{aligned}\frac{1}{d(t)} \sum_{u\in\Gamma(t)} \phi_{st}(u) &= \frac{2m}{\kappa(s,t)d(t)} \\ \Leftrightarrow \frac{1}{R_{st}} &= \frac{2m}{\kappa(s,t)} \\ \Leftrightarrow \kappa(s,t) &= 2mR_{st}\end{aligned}$$

So $\kappa_G(s,t) = 2mR_{st,G}$ and $\kappa_{G'}(s,t) = 2(m+1)R_{st,G'}$. By the Shorting Law [1, p. 75] or equivalently, Rayleigh's Monotonicity Law [1, p. 54], adding an edge to G does not increase any resistance R_{st}. So $R_{st,G'} \leq R_{st,G}$ and hence

$$\kappa_{G'}(i,j) = 2(m+1)R_{st,G'} \leq 2(m+1)R_{st,G} = \frac{2m(m+1)}{m} R_{st,G} = \left(\frac{m+1}{m}\right) \kappa_G(i,j)$$

4 Metropolis Filter

The Metropolis filter is an algorithm that computes a set of transition probabilities so the random walk converges to a given probability distribution. Let $G = (V,E)$ be a graph and assume it is d-regular (a regular graph with vertices of degree d). Let $F : V \to \mathbb{R}_+$ be any function. Let v_t be the node reached after t steps, and let u be a neighbour of v_t. We design the transition probabilities as follows. If $\frac{F(u)}{F(v_t)} \geq 1$, then move to u. Else, move to u with probability $\frac{F(u)}{F(v_t)}$, and stay at v_t with probability $1 - \frac{F(u)}{F(v_t)}$. In other words,

$$\begin{aligned}&\text{If } \frac{F(u)}{F(v_t)} \geq 1, \text{ then } v_{t+1} = u \\ &\text{If } \frac{F(u)}{F(v_t)} < 1, \text{ then } v_{t+1} = \begin{cases} u, & \text{w.p. } \frac{F(u)}{F(v_t)} \\ v_t, & \text{w.p. } 1 - \frac{F(u)}{F(v_t)} \end{cases}\end{aligned}$$

To compute the transition probabilities, let $U_n \sim Uniform[0,1]$. Note that

$$\begin{aligned}&\text{If } \frac{F(u)}{F(v_t)} \geq 1, \text{ then } P\left(U_n < \frac{F(u)}{F(v_t)}\right) = 1 \\ &\text{If } \frac{F(u)}{F(v_t)} < 1, \text{ then } P\left(U_n < \frac{F(u)}{F(v_t)}\right) = \frac{F(u)}{F(v_t)}\end{aligned}$$

Hence

$$p_{v_t,u} = min\left\{1, \frac{F(u)}{F(v_t)}\right\}, \quad p_{v_t,v_t} = 1 - \sum_{v_t u\in E} p_{v_t,u}$$

Now consider the probability distribution

$$Q_F(v) = \frac{F(v)}{\sum_{w \in V} F(w)}$$

This Markov chain is time reversible w.r.t $Q_F(v)$

$$\begin{aligned}
Q_F(v_t)p_{v_t,u} &= \frac{F(v_t)}{\sum F(w)} min\left\{1, \frac{F(u)}{F(v_t)}\right\} = min\left\{\frac{F(v_t)}{\sum F(w)}, \frac{F(u)}{\sum F(w)}\right\} = min\{F(v_t), F(u)\} \\
Q_F(u)p_{u,v_t} &= \frac{F(u)}{\sum F(w)} min\left\{1, \frac{F(v_t)}{F(u)}\right\} = min\left\{\frac{F(u)}{\sum F(w)}, \frac{F(v_t)}{\sum F(w)}\right\} = min\{F(u), F(v_t)\}
\end{aligned}$$

Since $Q_F(v_t)p_{v_t,u} = Q_F(u)p_{u,v_t}$, Q_F is the stationary distribution.

Remark: In class, we started with the state space $S = \mathbb{Z}$ and a given probability distribution $\{\pi_i\}$ on S. Then we created a Markov chain with transition probabilities $\{p_{ij}\}$ s.t. π_i is the stationary distribution. In the above example, the algorithm is very similar, except we start with a function $F : V \to \mathbb{R}_+$ instead of a probability distribution. We obtain the stationary distribution by normalizing i.e. dividing by $\sum_{w \in V} F(w)$.

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A Another Approach to Time Reversibility

The following example serves as a motivation for defining time reversibility [6]. We start with a Markov chain $\{X_n : n \in \mathbb{N}\}$ on some state space S, transition matrix P and stationary distribution π. Let $\{X_n^* : n \in \mathbb{N}\}$ be a stationary version of the chain, where $X_0^* \sim \pi$. Then $X_n^* \sim \pi$ for all $n \in \mathbb{N}$.

Using notation from class, let $\mu_i^{(n)} = P(X_n = i)$, and $\mu^{(n)} = \left(\mu_1^{(n)}, \mu_2^{(n)}, ...\right)$. By induction, we showed that $\mu^{(n)} = \mu^{(0)} P^n$ [5, p. 4]. In the stationary case, $\mu_i^{(0)} = \pi_i$, so $\mu^{(0)} = \pi$, and so $\mu^{(n)} = \mu^{(0)} P^n = \pi$, using the property that $\pi P = \pi$.

Now we extend $\{X_n^* : n \in \mathbb{N}\}$ to $\{X_n^* : n \in \mathbb{Z}\}$ by shifting the origin into the past.

Define $X_n^*(k) = X_{n+k}^*$ for $-k \le n < \infty$.

$$\begin{aligned}
&X_{-k}^*(k) = X_0^* \\
&X_{-k+1}^*(k) = X_1^* \\
&\vdots
\end{aligned}$$

So we get a new Markov chain $\{X_n^*(k) : -k \le n < \infty, n \in \mathbb{Z}\}$. Note that $X_n^*(k) = X_{n+k}^* \sim \pi$, so the new Markov chain is also stationary. Now let $k \to \infty$ to get $\{X_n^* : n \in \mathbb{Z}\}$ which is stationary.

Next we look at the Markov chain $\{X_n^* : n \in \mathbb{Z}\}$ in reverse time, and compute its transition probabilities.

Let $X_n^{(r)} = X_{-n}^*$ for $n \in \mathbb{N}$. Then $\{X_n^{(r)} : n \in \mathbb{N}\}$ is still a Markov chain by the Markov property that given the present state, past and future states are independent. Let $\{p_{ij}^{(r)}\}$ be the transition probabilities for $\{X_n^{(r)}\}$. Then

$$\begin{aligned}
p_{ij}(r) &= P(X_1^{(r)} = j | X_0^{(r)} = i) \\
&= P(X_{-1}^* = j | X_0^* = i) \\
&= P(X_0^* = j | X_1^* = i) \\
&= P(X_1^* = i | X_0^* = j) \frac{P(X_0^* = j)}{P(X_1^* = i)} \\
&= p_{ji} \frac{\pi_j}{\pi_i}
\end{aligned}$$

Now we can define a Markov chain as time reversible if $\{X_n^{(r)}\}$ has the same transition probabilities as $\{X_n^*\}$, i.e. $p_{ij}(r) = p_{ij}$ $\forall i, j \in S$. Then we arrive at the reversible property $\pi_i p_{ij} = \pi_j p_{ji}$.

B Section 3.1 Claim 2

We show $H(i,k) = k^2 - i^2$.

$$\begin{aligned}
H(i,k) &= H(i,k-1) + 2k - 1 = (H(i,k-2) + 2(k-1) - 1) + 2k - 1 \\
&= H(i,k-2) + (2(k-1) - 1) + (2k-1) \\
&\ \vdots \\
&= \sum_{j=i+1}^{k} 2j - 1 = \left(\sum_{j=i+1}^{k} 2j\right) - (k-i) = 2\left(\sum_{j=1}^{k} j - \sum_{j=1}^{i} j\right) - (k-i) \\
&= 2\left(\frac{k(k+1)}{2} - \frac{i(i+1)}{2}\right) - (k-i) = k^2 + k - i^2 - i - k + i \\
&= k^2 - i^2
\end{aligned}$$

C Section 3.2 Claim 2

We have the recurrent relation $f(n) = f(n-1) + (n-1)$, so $f(n) - f(n-1) = n-1$. Note that $f(1) = 0$. Then

$$f(n) = \sum_{i=2}^{n} f(i) - f(i-1) = \sum_{i=2}^{n} i - 1 = \sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}$$

D Section 3.3 Claim 1

We show $\sum_{t=1}^{\infty} t\left(\frac{n-2}{n-1}\right)^{t-1} \frac{1}{n-1} = n-1$. Let $x = \frac{n-2}{n-1}$. Note that $|x| < 1$.

$$\begin{aligned}
\sum_{t=1}^{\infty} t\left(\frac{n-2}{n-1}\right)^{t-1} \frac{1}{n-1} &= \frac{1}{n-1}\sum_{t=1}^{\infty} t x^{t-1} = \frac{1}{n-1}\left(\sum_{t=1}^{\infty} x^t\right)' \\
&= \frac{1}{n-1}\left(\frac{1}{1-x}\right)' = \frac{1}{n-1}\left(\frac{1}{(1-x)^2}\right) \\
&= \frac{1}{n-1}\left(\frac{1}{1-2x+x^2}\right) = \frac{1}{n-1}\left(\frac{1}{1-2(\frac{n-2}{n-1}) + \frac{(n-2)^2}{(n-1)^2}}\right) \\
&= \frac{1}{n-1}\left(\frac{(n-1)^2}{(n-1)^2 - 2(n-1)(n-2) + (n-2)^2}\right) \\
&= \frac{n-1}{((n-1)-(n-2))^2} \\
&= n-1
\end{aligned}$$

E Section 3.3 Claim 2

We show $E(\tau_{i+1}-\tau_i)=\frac{n-1}{n-i}$. Proof is very similar to above proof for Section 3.3 Claim 1.

$$\begin{aligned}
E(\tau_{i+1}-\tau_i) &= \sum_{t=1}^{\infty} tP(\tau_{i+1}-\tau_i=t) = \sum_{t=1}^{\infty} t\left(\frac{i-1}{n-1}\right)^{t-1}\frac{n-i}{n-1} \qquad \text{let } x=\frac{i-1}{n-1} \\
&= \frac{n-i}{n-1}\sum_{t=1}^{\infty} tx^{t-1} = \frac{n-i}{n-1}\left(\sum_{t=1}^{\infty} x^t\right)' \\
&= \frac{n-i}{n-1}\left(\frac{1}{1-2x+x^2}\right) = \frac{n-i}{n-1}\left(\frac{1}{1-2(\frac{i-1}{n-1})+\frac{(i-1)^2}{(n-1)^2}}\right) \\
&= \frac{n-i}{n-1}\left(\frac{(n-1)^2}{(n-1)^2-2(i-1)(n-1)+(i-1)^2}\right) \\
&= \frac{(n-i)(n-1)}{((n-1)-(i-1))^2} \\
&= \frac{n-1}{n-i}
\end{aligned}$$

Next, we show $\sum_{i=1}^{n-1}\frac{n-1}{n-i}\approx n\log n$.

$$\sum_{i=1}^{n-1}\frac{n-1}{n-i} = (n-1)\sum_{i=1}^{n-1}\frac{1}{n-i} = (n-1)\sum_{k=1}^{n-1}\frac{1}{k}$$

Now we use some calculus to show $\sum_{k=1}^{n}\frac{1}{k}\approx\log n$ [2]. Note that

$$\sum_{x=1}^{n}\frac{1}{x+1} < \int_1^n \frac{1}{x}dx = \log n < \sum_{x=1}^{n}\frac{1}{x}$$

Let $H_n=\sum_{x=1}^{n}\frac{1}{x}$ so we get

$$\begin{aligned}
H_n - 1 < \log n < H_n \\
0 < H_n - \log n < 1
\end{aligned}$$

Let $\delta_n = H_n - \log n$. Then

$$\begin{aligned}
\delta_n - \delta_{n+1} &= (H_n-\log n)-(H_{n+1}-\log(n+1)) \\
&= \log(n+1)-\log n + (H_n - H_{n+1}) \\
&= \int_n^{n+1}\frac{1}{x}dx - \frac{1}{n+1} \\
&> 0
\end{aligned}$$

Since δ_n is bounded and monotone decreasing, it converges by the Monotone Convergence Theorem. The value γ to which it converges is called the Euler-Mascheroni Constant, which is approximately 0.5772. So for large n we can approximate H_n by $\log n+\gamma$.

Now back to the original question

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{n-1}{n-i} &= (n-1)\sum_{k=1}^{n-1}\frac{1}{k} \\ &\approx (n-1)(\log(n-1)+\gamma) \\ &\approx n\log n\end{aligned}$$

F Section 3.4.1 Claim 1

We show

$$\begin{bmatrix}(n-4)/(n-2) & 2/(n-2) & 0 & 0 \\ (n-2)/n & 0 & 2/n & 0 \\ 0 & (n-2)/n & 0 & 2/n \\ 0 & 0 & 2/(n-2) & (n-4)/(n-2)\end{bmatrix}$$

Suppose we start at state 1, specifically at some node i. Then $p_{12} = p_{iu} = \frac{1}{n/2-1} = \frac{2}{n-2}$. Clearly, we cannot go from state 1 to 3 or 1 to 4 in one step, so $p_{13} = p_{14} = 0$ and $p_{11} = 1 - \frac{2}{n-2}$. Next, $p_{23} = p_{uv}$ and as shown before, this is $\frac{2}{n}$ and so $p_{21} = 1 - \frac{2}{n}$. For p_{34}, we can apply Gambler's Ruin to the path from u to j. Using notation from class, $u = 0, a = v = 1, c = j = \frac{n}{2}$. Then $p_{34} = P_a(T_c < T_0) = \frac{a}{c} = \frac{1}{n/2} = \frac{2}{n}$ [5, p. 29]. For p_{43}, this is essentially $P_j(T_v < T_j)$. But from node j, we can only go to the node to its left. Call this node $j-1$. Then $p_{43} = P_{j-1}(T_v < T_j)$. Again, apply Gambler's ruin to the path from v to j, with $v = 0, a = j-1 = \frac{n}{2} - 2, c = j = \frac{n}{2} - 1$. Then $P_{j-1}(T_v < T_j) = P_a(T_0 < T_c) = 1 - P_a(T_c < T_0) = 1 - \frac{a}{c} = 1 - \frac{n/2-2}{n/2-1} = 1 - \frac{n-4}{n-2}$.

Note that while the probability p_{34} is defined, the random walk cannot actually jump from state 3 to 4 in one step. Since we let state 3 be node v and 4 be node j, there are still $\frac{n}{2} - 2$ nodes in between. So the expected time to reach state 4 from 3 is in fact larger than one would expect.

For the alternate case where we let $p_{41} = 1$ and $p_{4i} = 0$ for $i \neq 1$, the transition probability matrix is then

$$\begin{bmatrix}(n-4)/(n-2) & 2/(n-2) & 0 & 0 \\ (n-2)/n & 0 & 2/n & 0 \\ 0 & (n-2)/n & 0 & 2/n \\ 1 & 0 & 0 & 0\end{bmatrix}$$

To compute the stationary probabilities π_i, we use the properties $\sum_{i\in S}\pi_i p_{ij} = \pi_j$ and $\sum_{i\in S}\pi_i = 1$.

$$\begin{aligned}
&\pi_4 = \pi_1 p_{14} + \pi_2 p_{24} + \pi_3 p_{34} + \pi_4 p_{44} = \left(\frac{2}{n}\right)\pi_3 \\
&\pi_3 = \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33} + \pi_4 p_{43} = \left(\frac{2}{n}\right)\pi_2 \\
&\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} + \pi_4 p_{41} = \left(\frac{n-4}{n-2}\right)\pi_1 + \left(\frac{n-2}{n}\right)\pi_2 + \pi_4 \\
&\quad \Rightarrow \left(\frac{2}{n-2}\right)\pi_1 = \left(\frac{n-2}{n}\right)\pi_2 + \pi_4 \Rightarrow \pi_1 = \left(\frac{(n-2)^2}{2n}\right)\pi_2 + \left(\frac{n-2}{2}\right)\pi_4 \\
&\pi_2 = \pi_1 p_{12} + \pi_2 p_{22} + \pi_3 p_{32} + \pi_4 p_{42} = \left(\frac{2}{n-2}\right)\pi_1 + \left(\frac{n-2}{n}\right)\pi_3
\end{aligned}$$

Rearrange to get

$$\pi_4 = \left(\frac{4}{n^2}\right)\pi_2, \quad \pi_3 = \left(\frac{2}{n}\right)\pi_2$$

$$\pi_1 = \left(\frac{(n-2)^2}{2n}\right)\pi_2 + \left(\frac{n-2}{2}\right)\pi_4 = \left(\frac{(n-2)^2}{2n}\right)\pi_2 + \left(\frac{4(n-2)}{2n^2}\right)\pi_2 = \left(\frac{n(n-2)^2+4(n-2)}{2n^2}\right)\pi_2$$

Then

$$\begin{aligned} 1 = \sum_{i \in S} \pi_i &= \left(\frac{n(n-2)^2+4(n-2)}{2n^2} + 1 + \frac{2}{n} + \frac{4}{n^2}\right)\pi_2 \\ &= \left(\frac{n(n-2)^2+4(n-2)+2n^2+4n+8}{2n^2}\right)\pi_2 \end{aligned}$$

And so

$$\begin{aligned} \pi_2 &= \frac{2n^2}{(n(n-2)^2+2n^2+8n)} = \frac{2n^2}{n^3-4n^2+4n+2n^2+8n} = \frac{2n^2}{n^3-2n^2+12n} \\ \pi_4 &= \left(\frac{4}{n^2}\right)\pi_2 = \frac{8}{n^3-2n^2+12n} \end{aligned}$$

G Section 3.4.2 Claim 1

Note that

$$\sum_{c=1}^{\infty}(2c)\frac{1}{2^c} = \sum_{c=1}^{\infty} c\left(\frac{1}{2}\right)^{c-1}$$

Again, using techniques from Appendices D and E, let $|x| < 1$. Then

$$\sum_{c=1}^{\infty} cx^{c-1} = \left(\sum_{c=0}^{\infty} x^c\right)' = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

So

$$\sum_{c=1}^{\infty} c\left(\frac{1}{2}\right)^{c-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$$