Errata for FIRST edition of "A First Look at Rigorous Probability"

NOTE: the corrections below (plus many more improvements) were all encorporated into the second edition:

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Errata for Third Printing, 2005:

- In Exercise 2.7.8, condition (ii) refers to *finite* intersections only.
- Exercise 14.4.8 requires an additional assumption, and is not correct as stated.

Errata to Second Printing, 2003 (already corrected in Third Printing):

[With thanks to Samuel Hikspoors, Bin Li, Mahdi Lotfinezhad, Ben Reason, Jay Sheldon, and Zemei Yang.]

- p. 18, Exercise 2.7.1: The phrase "together with the singleton set $\{0\}$" should be at the end of the first sentence, not the second. That is, $\mathcal{J}'$ also includes the singleton set $\{0\}$.
- p. 30, Exercise 3.6.8, condition (i) should read: "(i) $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are independent provided that $i_{j+1} \geq i_j + 2$ for $1 \leq j \leq k-1$".
- p. 41, Exercise 4.5.7: It should be assumed that X and Y are *independent*.
- p. 41, Exercise 4.5.9(c): append the words "with finite means".
- p. 123, Exercise 12.1.7: For definiteness, we should assume that $\Omega = \mathbf{R}$ (together with the Borel subsets).
- p. 127, lines 3 and 5: These calculations implicitly assume that $S \subseteq [0,2]$. So, to make them valid, we should either explicitly assume that $S \subseteq [0,2]$, or replace S by $S \cap [0,2]$ where required.
- p. 138, Exercise 14.4.2: Assume $\mathbf{E}[Y_n] < \infty$ for all n.
- p. 138, Exercise 14.4.3: Assume $\mathbf{E}(|\phi(X_n)|) < \infty$ for all n. Also, you may assume the *conditional* form of Jensen's inequality, i.e. that $\mathbf{E}[\phi(X) \,|\, \mathcal{G}] \geq \phi(\mathbf{E}[X \,|\, \mathcal{G}])$.
- p. 138, Exercise 14.4.5 is awkwardly written; a better version is:
  "Let $\{X_n\}$ be simple symmetric random walk on the integers, with $X_0 = 0$. Let $\tau = \inf\{n \geq 5 : X_{n+1} = X_n + 1\}$ be the first time after 4 which is just before the

chain increases. Let $\rho = \tau + 1$.
(a) Is τ a stopping time? Is ρ a stopping time?
(b) Use Theorem 14.1.3 to compute $\mathbf{E}[X_\rho]$.
(c) Use the result of part (b) to compute $\mathbf{E}[X_\tau]$."

- Index: Add entry "Fatou's Lemma, 86".

Errata to First Printing, 2000 (already corrected in Second Printing):

[With thanks to Tom Baird, Meng Du, Avery Fullerton, Longhai Li, Hadas Moshonov, Nataliya Portman, and Idan Regev.]

- p. 4, replace Exercise 1.3.2 by: "Suppose $\Omega = \{1, 2, 3\}$ and $\mathcal{F}$ is the collection of all subsets of Ω. Find (with proof) necessary and sufficient conditions on the real numbers x, y, and z, such that there exists a countably additive probability measure $\mathbf{P}$ on $\mathcal{F}$, with $x = \mathbf{P}\{1, 2\}$, $y = \mathbf{P}\{2, 3\}$, and $z = \mathbf{P}\{1, 3\}$."

- pp. 9–10, in Exercise 2.3.2 parts (d) and (e), "$\leq$" should be "$\geq$". Also, in parts (c), (d), and (e), the word "disjoint" should be omitted.

- p. 11, line 5: "$B_n \subseteq C_{nk}$" should be "$B_n \subseteq \bigcup_k C_{nk}$".

- p. 13, line 10 from bottom: "since $\mathbf{P} \leq \mathbf{P}^*$ on $\mathcal{F}_0$" should be "since $\mathbf{P}^* \leq \mathbf{P}$ on $\mathcal{F}_0$"

- p. 17, line 4, expand "then by additivity ..." to "then $\mathcal{B}_0$ is an algebra, and by additivity ...".

- p. 17 middle, "since $B_n \in \mathcal{J}$" should be "since $B_n \in \mathcal{B}_0$". Also, "A_n" should be "A_i" (four times).

- p. 18, Exercise 2.7.1, expand "all finite disjoint unions of elements of $\mathcal{J}'$." to "all finite disjoint unions of elements of $\mathcal{J}'$, together with the single set $\{0\}$."

- p. 19, Exercise 2.7.4: "$P(A)$" should be "$\mathbf{P}(A)$" (twice).

- p. 19, Exercise 2.7.7: "$\mathbf{P}\{1\} = \frac{1}{3}, \mathbf{P}\{2\} = \frac{2}{3}$" should be "$\mathbf{P}_2\{1\} = \frac{1}{3}, \mathbf{P}_2\{2\} = \frac{2}{3}$".

- p. 24, the end of the proof of Proposition 3.3.1, replace "$\ldots = \lim_{n\to\infty} \mathbf{P}(A)$, where ... nested sequence." by "$\ldots = \lim_{n\to\infty} \mathbf{P}(A_n)$, where the last equality is the only time we use that the $\{A_m\}$ are a nested sequence."

- p. 26, final displayed equation: Final sum should be $\sum_{k=m}^{\infty} \mathbf{P}(A_k)$.

- p. 27 bottom, "$B_n = \{r_{n+1} = r_{n+2} = \ldots = r_{n+[\log_2 \log_2 n]}\} = 1$" should be "$B_n = \{r_{n+1} = r_{n+2} = \ldots = r_{n+[\log_2 \log_2 n]} = 1\}$".

- p. 32 bottom, in definition of Y: "irrational" should be "rational".

- p. 41, Exercise 4.5.9(a): "$Z^+ - Z^-$" should be "Z^+ and Z^-".
- p. 49 eqn. (5.3.9), "$k+1$" should be "$\ell+1$".
- p. 49 eqn. (5.3.11), "u_k" should be "k".
- p. 49, lines 4–6 from bottom, replace "Hence, for all sufficiently large n we have ... for all sufficiently large k." by "Hence, for any $\alpha > 1$ and $\delta > 0$, with probability 1 we have $m/(1+\delta)\alpha \leq \frac{S_k}{k} \leq (1+\delta)\alpha m$ for all sufficiently large k."
- p. 50, Exercise 5.4.1: "variables" should be "variable".
- p. 65, Exercise 7.4.3, Hint: Omit the word "two".
- pp. 65, 66, 68, and 72: "Subsection 7.2.0" should be "Subsection 7.2".
- p. 69, line 8 from bottom: "Thus" should be "This".
- p. 72, line 4: "$X_n = j$" should be "$X_n = i$".
- p. 74 towards bottom, "$\binom{n}{i}\frac{1}{2^d}$" should be "$\binom{d}{i}\frac{1}{2^d}$".
- p. 75, Definition 8.3.3: "Give" should be "Given".
- p. 76, line 12: "divisor of $\{n \in \mathbf{N};\ p_{jj}^{(n)}\}$" should be "divisor of $\{n \in \mathbf{N};\ p_{jj}^{(n)} > 0\}$".
- p. 78, line 5: "indeed, we have $p_{(ij),(k\ell)} > 0$" should be "indeed, we have $p_{(ij),(k\ell)}^{(n)} > 0$".
- p. 83, line 10, Exercise 8.5.2: "for all states i and j" should be "for some states i and j".
- p. 83, Exercise 8.5.3: replace "with distinct states i and j" by "and some distinct states i and j".
- p. 86, line 8 from bottom, "bounded convergence theorem" should be "monotone convergence theorem".
- p. 90 middle: after "repeatedly apply Proposition 9.2.1", add in parentheses "(or use Proposition 9.3.2 below)".
- p. 100, Exercise 10.1.2: Replace "$\mu(A) = \int f\,\mathbf{1}_A\,d\lambda$" by "$\mu(A) = \int_0^1 f\,\mathbf{1}_A\,d\lambda$".
- p. 100, Exercise 10.1.4: Replace the last sentence by "Construct a sequence $\{\mu_n\}$ of probability measures, each absolutely continuous with respect to Lebesgue measure, such that $\mu_n \Rightarrow \mu$."
- p. 100: Omit Exercise 10.1.7.
- p. 102 middle, "Like for characteristic functions, ..." should be "Like for moment generating functions, ...".

- p. 103, line 4 from bottom, the two consecutive factors of "$\left|\frac{e^{-ita}-e^{-itb}}{it}\phi(t)\right|$" should both be omitted.

- p. 104 middle, "$\int_0^\infty e^{ux}du$" should be "$\int_0^\infty e^{-ux}du$".

- p. 110, line 9 from bottom: "$\sqrt{2}$" should be "$\sqrt{n}$", and the entire expansion should be raised to the power n.

- p. 112, lines 8 and 11: "$x \to \infty$" should be "$x \to -\infty$" (twice).

- p. 112, line 6 from bottom: add extra closing parenthesis just before "$\Rightarrow$" symbol.

- p. 114, line 4: "$(-\infty, R)$" should be "$(-\infty, -R)$".

- p. 123, Exercise 12.1.2 (a), "Subsection 9.4.0" should be "Subsection 9.4".

- p. 127, line 6 from bottom: "$\mathbf{E}(Y \mid \mathcal{G}) = \mathbf{E}(X)$" should be "$\mathbf{E}(Y \mid \mathcal{G}) = \mathbf{E}(Y)$".

- p. 128, line 8: for clarity, "out of the conditioning" should be "out of the conditional expectation".

- p. 129, Exercise 13.1.1: Expand "where $f : \mathbf{R}^2 \to \mathbf{R}$ is" to "where $dx\,dy$ is two-dimensional Lebesgue measure, and where $f : \mathbf{R}^2 \to \mathbf{R}$ is".

- p. 133, line 10: "... extremely high that $\tau \leq 10^{12}$" should be "... extremely high that $\tau < 10^{12}$". In the following line, "the rare case that $\tau > 10^{12}$" should be "the rare case that $\tau = 10^{12}$".

- p. 133, end of Subsection 14.1: Add Addendum (at end).

- p. 135, end of proof of Lemma 14.2.3: Add "Finally, note that replacing $\{X_n\}$ by $\{\max(X_n, \alpha)\}$ can only *decrease* $|X_M - X_0|$, so the inequality still holds as written."

- p. 137, line 4 from bottom: "$\mathbf{E}(X^2) = \ldots$" should be "$\mathrm{Var}(X) = \ldots$"

- p. 144, Exercise 15.2.6(b), "Related this" should be "Relate this".

- p. 149, Exercise 15.4.3: Replace "$|f(x)-f(y)| \leq |x-y|$" by "$|f(x)-f(y)| \leq \alpha|x-y|$".

- p. 151, Exercise 15.6.4: "$\pi(x) \geq 0$" should be "$\pi(x) > 0$".

- p. 156, line 2: "assume" should be "assumed".

- On p. 157 bottom, after "Recall that here r is the risk-free interest rate", add "and σ is the volatility".

- p. 161, lines 4 and 5: "it's" should be "its" (twice).

- p. 161, middle, in definition of $\mathbf{Q}$: "$m \neq 0$" should be "$n \neq 0$".

- p. 171, Index, add reference to page 22 for "Borel-measurable".

ADDENDUM: Extra material for end of Section 14.1, on page 133:

(Already included in Second and Third Printings.)

Theorem 14.1.3. Let $\{X_n\}_{n=0}^{\infty}$ be a martingale with stopping time τ. Suppose $\mathbf{P}(\tau < \infty) = 1$, and $\mathbf{E}|X_\tau| < \infty$, and $\lim_{n\to\infty} \mathbf{E}[X_n \, \mathbf{1}_{\tau>n}] = 0$. Then $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$.

Proof. Let $Z_n = X_{\min(\tau,n)}$ for $n = 0, 1, 2, \ldots$. Then $Z_n = X_\tau \mathbf{1}_{\tau \le n} + X_n \mathbf{1}_{\tau>n} = X_\tau - X_\tau \mathbf{1}_{\tau>n} + X_n \mathbf{1}_{\tau>n}$, so $X_\tau = Z_n - X_n \mathbf{1}_{\tau>n} + X_\tau \mathbf{1}_{\tau>n}$. Hence,

$$\mathbf{E}[X_\tau] = \mathbf{E}[Z_n] - \mathbf{E}[X_n \mathbf{1}_{\tau>n}] + \mathbf{E}[X_\tau \mathbf{1}_{\tau>n}]\,.$$

Since $\min(\tau, n)$ is a bounded stopping time, $\mathbf{E}[Z_n] = \mathbf{E}[X_0]$ for all n by Corollary 14.1.2. As $n \to \infty$, the second term goes to 0 by assumption. Also, the third term goes to 0 by the Dominated Convergence Theorem, since $E[X_\tau] < \infty$, and $\mathbf{1}_{\tau>n} \to 0$ w.p. 1 since $\mathbf{P}[\tau < \infty] = 1$. Hence, letting $n \to \infty$, we obtain that $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$. ∎

Corollary 14.1.4. Let $\{X_n\}_{n=0}^{\infty}$ be a martingale with stopping time τ, such that $\mathbf{P}[\tau < \infty] = 1$. Assume $|X_n| \le M$ whenever $n \le \tau$, for all n and some fixed $M < \infty$. Then $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$.

Proof. Clearly $|X_\tau| \le M$, so that $\mathbf{E}|X_\tau| \le M < \infty$. Also $|\mathbf{E}(X_n \mathbf{1}_{\tau>n})| \le \mathbf{E}(|X_n| \mathbf{1}_{\tau>n}) \le M\,\mathbf{P}(\tau > n)$, which converges to 0 as $n \to \infty$ since $\mathbf{P}[\tau < \infty] = 1$. Hence, the result follows from Theorem 14.1.3. ∎

Exercise 14.1.5. Let $0 < a < c$ be integers. Let $\{X_n\}$ be simple symmetric random walk (i.e., with parameter $p = 1/2$), started at $X_0 = a$. Let $\tau = \inf\{n \ge 1;\ X_n = 0 \text{ or } c\}$.
(a) Prove that $\{X_n\}$ is a martingale.
(b) Prove that $\mathbf{E}[X_\tau] = a$. [Hint: Use Corollary 14.1.4.]
(c) Use this fact to derive an alternative proof of the gambler's ruin formula given in Section 7.2, for the case $p = 1/2$.

Exercise 14.1.6. Let $0 < p < 1$ with $p \ne 1/2$, and let $0 < a < c$ be integers. Let $\{X_n\}$ be simple random walk with parameter p, started at $X_0 = a$. Let $\tau = \inf\{n \ge 1;\ X_n = 0 \text{ or } c\}$. Let $Z_n = ((1-p)/p)^{X_n}$ for $n = 0, 1, 2, \ldots$.
(a) Prove that $\{Z_n\}$ is a martingale.
(b) Prove that $\mathbf{E}[Z_\tau] = ((1-p)/p)^a$. [Hint: Use Corollary 14.1.4.]
(c) Use this fact to derive an alternative proof of the gambler's ruin formula given in Section 7.2, for the case $p \ne 1/2$.