WEB APPENDIX FOR:

The Coupling/Minorization/Drift Approach to Markov Chain Convergence Rates

by (in alphabetical order)

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This web appendix provides proofs of the computational lemmas in the main article, which is available at: www.probability.ca/NoticesArt.pdf

Proof of Lemma 1:

To avoid problematic configurations where the particles are very close together, we first set $\mathcal{X}' = \{(x_1, x_2, x_3) \in \mathcal{X} : \forall 1 \le i < j \le 3, |x_i - x_j| \ge 1/4\}$. Since $\mathcal{X}'$ is a compact set, and π is continuous and positive on $\mathcal{X}'$, it must achieve its minimum $m := \min_{x,y \in \mathcal{X}'} \frac{\pi(y)}{\pi(x)} > 0$ there. Let $A \subset \mathcal{X}$. Then from any state $x \in \mathcal{X}$, the chain will move into A on the next step provided that the proposed new configuration y is within the subset A, and that the proposal is accepted. Hence,

$$P(x, A) \;=\; \int_A P(x, dy) \;=\; \int_A \min[1,\; \frac{\pi(y)}{\pi(x)}]\, dy \;\ge\; \int_{A \cap \mathcal{X}'} m\, dy \;=\; m\, \mathrm{Leb}(A \cap \mathcal{X}')\,,$$

where Leb is Lebesgue measure on $\mathbb{R}^6$. So, if we set $\epsilon = m\,\mathrm{Leb}(\mathcal{X}')$, and $\nu(A) = \mathrm{Leb}(A \cap \mathcal{X}') \,\big/\, \mathrm{Leb}(\mathcal{X}')$, then $\epsilon > 0$, and ν is a probability measure, and $P(x, A) \ge \epsilon\, \nu(A)$, i.e. a uniform minorization condition is satisfied.

To obtain quantitative convergence bounds, we need to estimate $\mathrm{Leb}(\mathcal{X}')$ and m. In order for $(x_1, x_2, x_3) \in \mathcal{X}'$, we can choose any $x_1 \in [0,1]^2$ (with two-dimensional area 1), then choose any $x_2 \in [0,1]^2 \setminus B(x_1, 1/4)$ (with area $\ge 1 - 3.14(1/4)^2$), then choose any $x_3 \in [0,1]^2 \setminus (B(x_1, 1/4) \cup B(x_2, 1/4))$ (with area $\ge 1 - 3.14(1/4)^2 - 3.14(1/4)^2$). [Here $B(x, r)$ is the two-dimensional disc centered at x of radius r, with area $3.14\, r^2$, where we write the constant as "3.14" to avoid confusion with the stationary distribution $\pi(\cdot)$.] Hence, $\mathrm{Leb}(\mathcal{X}') \ge (1)(1 - \frac{3.14}{16})(1 - \frac{3.14}{8}) \ge 0.48$.

Also, for any $x \in \mathcal{X}'$, we must have $0 \le |x_i| \le \sqrt{2}$ and $1/4 \le |x_i - x_j| \le \sqrt{2}$, so therefore

$$0 \;\le\; |x_1| + |x_2| + |x_3| \le 3\sqrt{2}\,, \quad \text{and} \quad \frac{3}{\sqrt{2}} \;\le\; \sum_{i<j} |x_i - x_j|^{-1} \le 12\,.$$

It follows that

$$m \;\geq\; \frac{e^{-C(3\sqrt{2})-D(12)}}{e^{-C(0)-D(3/\sqrt{2})}} \;=\; e^{-C(3\sqrt{2})-D(12-(3/\sqrt{2}))} \;\geq\; e^{-C(4.25)-D(9.88)}\,.$$

Hence,

$$\epsilon \;=\; m\,\mathrm{Leb}(\mathcal{X}') \;\geq\; (0.48)\, e^{-C(4.25)-D(9.88)}\,,$$

as claimed.

Proof of Lemma 2:

Let $x \in C$. Without loss of generality, assume $x \geq 0$. First consider $B \subset [-1, 1]$, and let $z \in [0,1]$ and $y \in B$. Then we must have $[0,1] \subseteq [x-2, x+2]$, and $B \subseteq [z-2, z+2]$. Hence, the proposal density q satisfies that $q(x,z) = q(z,y) = \frac{1}{4}$. Also, $\pi(x) \leq e^0 = 1$, and $e^{-1} \leq \pi(y) \leq 1$, and $\pi(z) \geq e^{-1}$, so if $\alpha(x,z) = \min[1, \frac{\pi(z)}{\pi(x)}]$ is the probability of accepting a proposed move from x to z, then $\alpha(x,z) \geq e^{-1}$ and $\alpha(z,y) \geq e^{-1}$. Hence,

$$P^2(x,B) \;\geq\; \int_B \int_{x-2}^{x+2} q(x,z)\,\alpha(x,z)\,q(z,y)\,\alpha(z,y)\,dz\,dy$$
$$\geq\; \int_B \int_0^1 (1/4)(e^{-1})(1/4)(e^{-1})\,dz\,dy \;=\; \frac{1}{16e^2}\,\mathrm{Leb}(B)\,.$$

Finally, for any $A \subseteq \mathbb{R}$,

$$P^2(x,A) \;\geq\; P^2(x,\, A \cap [-1,1]) \;\geq\; \frac{1}{16e^2}\,\mathrm{Leb}(A \cap [-1,1]) \;=\; \frac{1}{8e^2}\,\nu(A)\,,$$

which gives the result.

Proof of Lemma 3:

Without loss of generality, assume $x \geq 0$. Note that

$$PV(x) \;=\; \int_{x-2}^{x+2} q(x,y)\;[V(y)\alpha(x,y) + V(x)(1-\alpha(x,y))]\;dy\,.$$

We first compute the "top half" of this integral, where $x \leq y \leq x-2$. Here

$\alpha(x,y) = \frac{\pi(y)}{\pi(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y}$, and $q(x,y) = 1/4$, so

$$
\begin{aligned}
&\int_x^{x+2} q(x,y)\ [V(y)\alpha(x,y) + V(x)(1-\alpha(x,y))]\ dy \\
&= \int_x^{x+2} \frac{1}{4} e^{\frac{y}{2}} e^{x-y} dy + \int_x^{x+2} \frac{1}{4} e^{\frac{x}{2}} (1 - e^{x-y}) dy) \\
&= \frac{1}{4} e^x \int_x^{x+2} e^{-\frac{y}{2}} dy + \frac{1}{4} e^{\frac{x}{2}} (2) - \frac{1}{4} e^{\frac{3x}{2}} \int_x^{x+2} e^{-y} dy \\
&= \frac{1}{4} e^x [-2e^{-\frac{x+2}{2}} + 2e^{-\frac{x}{2}}] + \frac{1}{4} e^{\frac{x}{2}} (2) - \frac{1}{4} e^{\frac{3x}{2}} [-e^{-x-2} + e^{-x}] \\
&= \frac{1}{4} e^{\frac{x}{2}} (-2e^{-1} + 2 + 2 + e^{-2} - 1) \\
&= \frac{1}{4} (3 + e^{-2} - 2e^{-1}) V(x) \ \equiv \ \lambda_1 V(x) ,
\end{aligned}
$$

where $\lambda_1 = \frac{1}{4}(3 + e^{-2} - 2e^{-1}) \doteq 0.6$. Then we consider three different cases:

Case 1: $x \in (2, \infty) \not\subseteq C = [-2, 2]$. Then $\alpha(x,y) := \min\{1, \frac{e^{-|y|}}{e^{-|x|}}\} = 1$ for all $y \in [x-2, x)$, so

$$
\begin{aligned}
PV(x) &= \int_{x-2}^{x} q(x,y) V(y) dy + \lambda_1 V(x) = \frac{1}{4} \int_{x-2}^{x} e^{\frac{y}{2}} dy + \lambda_1 V(x) \\
&= \frac{1}{4} e^{\frac{x}{2}} 2(1 - e^{-1}) + \lambda_1 V(x) = (\frac{1}{2}(1 - e^{-1}) + \lambda_1) V(x) \le 0.916\, V(x) .
\end{aligned}
$$

Case 2: $x \in [1, 2] \subseteq C$. Again $\alpha(x,y) = 1$ for all $y \in [x-2, x]$, so

$$
\begin{aligned}
PV(x) &= \int_{x-2}^{x} V(y) q(x,y) dy + \lambda_1 V(x) = \frac{1}{4} (\int_{x-2}^{0} e^{-\frac{y}{2}} dy + \int_0^x e^{\frac{y}{2}} dy) + \lambda_1 V(x) \\
&= \frac{1}{4} (\int_0^{2-x} e^{\frac{y}{2}} dy + \int_0^x e^{\frac{y}{2}} dy) + \lambda_1 V(x) = \frac{1}{2} (e^{\frac{x}{2}} + e^{1-\frac{x}{2}}) - 1 + \lambda_1 e^{\frac{x}{2}}
\end{aligned}
$$

Let $z = e^{\frac{x}{2}}$. Then, computing numerically,

$$
\max_{x \in [1,2]} [PV(x) - 0.916 V(x)] \ = \ \max_{z \in [\sqrt{e}, e]} \left[\frac{1}{2} (z + \frac{e}{z}) - 1 + \lambda_1 z - 0.916\, z \right] \ \le \ 0.13 .
$$

Case 3: $x \in [0, 1] \subseteq C$. Then $\alpha(x,y) = 1$ for any $y \in [-x, x]$.

$$
\begin{aligned}
PV(x) = &\int_{x-2}^{-x} \Big[q(x,y)\alpha(x,y)V(y) + q(x,y)(1 - \alpha(x,y))V(x) \Big] dy \\
&+ \int_{-x}^{x} q(x,y) V(y)\, dy + \lambda_1 V(x) \\
= &\frac{1}{4} e^{\frac{x}{2}} \int_x^{2-x} (e^{\frac{x-y}{2}} + 1 - e^{x-y})\, dy + \frac{1}{2} \int_0^x e^{\frac{y}{2}} dy + \lambda_1 e^{\frac{x}{2}} \\
= &\frac{e^{\frac{x}{2}}}{4} \Big[-2e^{x-1} + e^{2(x-1)} - 2x + 3 \Big] + e^{\frac{x}{2}} - 1 + \lambda_1 e^{\frac{x}{2}} .
\end{aligned}
$$

Computing numerically, this implies that

$$\max_{x\in[0,1]} [PV(x) - 0.916\, V(x)] \;\le\; 0.285\,.$$

Combining these three cases (and their symmetric versions for $x < 0$) shows that the univariate drift condition

$$PV(x) \;\le\; 0.916\, V(x) + 0.285\, \mathbf{1}_C(x)$$

holds for all $x \in \mathcal{X}$, as claimed.