7

The Nonlinear State Space Model

In applying the results and concepts of Part I in the domains of times series or systems theory, we have so far analyzed only linear models in any detail, albeit rather general and multidimensional ones. This chapter is intended as a relatively complete description of the way in which nonlinear models may be analyzed within the Markovian context developed thus far. We will consider both the general nonlinear state space model, and some specific applications which take on this particular form.

The pattern of this analysis is to consider first some particular structural or stability aspect of the associated deterministic control, or CM(F), model and then under appropriate choice of conditions on the disturbance or noise process (typically a density condition as in the linear models of Section 6.3.2) to verify a related structural or stability aspect of the stochastic nonlinear state space NSS(F) model.

Highlights of this duality are

(i) if the associated CM(F) model is *forward accessible* (a form of controllability), and the noise has an appropriate density, then the NSS(F) model is a T-chain (Section 7.1);

(ii) a form of irreducibility (the existence of a *globally attracting state* for the CM(F) model) is then equivalent to the associated NSS(F) model being a ψ-irreducible T-chain (Section 7.2);

(iii) the existence of periodic classes for the forward accessible CM(F) model is further equivalent to the associated NSS(F) model being a periodic Markov chain, with the periodic classes coinciding for the deterministic and the stochastic model (Section 7.3).

Thus we can reinterpret some of the concepts which we have introduced for Markov chains in this deterministic setting; and conversely, by studying the deterministic model we obtain criteria for our basic assumptions to be valid in the stochastic case.

In Section 7.4.3 the adaptive control model is considered to illustrate how these results may be applied in specific applications: for this model we exploit the fact that $\boldsymbol{\Phi}$ is generated by a NSS(F) model to give a simple proof that $\boldsymbol{\Phi}$ is a ψ-irreducible and aperiodic T-chain.

We will end the chapter by considering the nonlinear state space model without forward accessibility, and showing how e-chain properties may then be established in lieu of the T-chain properties.

7.1 Forward Accessibility and Continuous Components

The nonlinear state space model NSS(F) may be interpreted as a control system driven by a noise sequence exactly as the linear model is interpreted. We will take such a viewpoint in this section as we generalize the concepts used in the proof of Proposition 6.3.3, where we constructed a continuous component for the linear state space model.

7.1.1 Scalar models and forward accessibility

We first consider the scalar model SNSS(F) defined by

$$X_n = F(X_{n-1}, W_n),$$

for some smooth (C^∞) function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and satisfying (SNSS1)-(SNSS2).

Recall that in (2.5) we defined the map F_k inductively, for x_0 and w_i arbitrary real numbers, by

$$F_{k+1}(x_0, w_1, \dots w_{k+1}) = F(F_k(x_0, w_1, \dots w_k), w_{k+1}),$$

so that for any initial condition $X_0 = x_0$ and any $k \in \mathbb{Z}_+$,

$$X_k = F_k(x_0, W_1, \dots, W_k).$$

Now let $\{u_k\}$ be the associated scalar "control sequence" for CM(F) as in (CM1), and use this to define the resulting state trajectory for CM(F) by

$$x_k = F_k(x_0, u_1, \dots, u_k), \qquad k \in \mathbb{Z}_+. \tag{7.1}$$

Just as in the linear case, if from each initial condition $x_0 \in \mathsf{X}$ a sufficiently large set of states may be reached from x_0, then we will find that a continuous component may be constructed for the Markov chain $\mathbf{X}$. It is not important that every state may be reached from a given initial condition; the main idea in the proof of Proposition 6.3.3, which carries over to the nonlinear case, is that the set of possible states reachable from a given initial condition is not concentrated in some lower dimensional subset of the state space.

Recall also that we have assumed in (CM1) that for the associated deterministic control model CM(F) with trajectory (7.1), the control sequence $\{u_k\}$ is constrained so that $u_k \in O_w$, $k \in \mathbb{Z}_+$, where the control set O_w is an open set in $\mathbb{R}$.

For $x \in \mathsf{X}$, $k \in \mathbb{Z}_+$, we define $A_+^k(x)$ to be the set of all states reachable from x at time k by CM(F): that is, $A_+^0(x) = \{x\}$, and

$$A_+^k(x) := \Big\{F_k(x, u_1, \dots, u_k) : u_i \in O_w, 1 \le i \le k\Big\}, \qquad k \ge 1. \tag{7.2}$$

We define $A_+(x)$ to be the set of all states which are reachable from x at some time in the future, given by

$$A_+(x) := \bigcup_{k=0}^{\infty} A_+^k(x) \tag{7.3}$$

The analogue of controllability that we use for the nonlinear model is called *forward accessibility*.

> **Forward accessibility**
>
> The associated control model CM(F) is called *forward accessible* if for each $x_0 \in \mathsf{X}$, the set $A_+(x_0) \subset \mathsf{X}$ has non-empty interior.

For general nonlinear models, forward accessibility depends critically on the particular control set O_w chosen. This is in contrast to the linear state space model, where conditions on the driving matrix pair (F, G) sufficed for controllability.

Nonetheless, for the scalar nonlinear state space model we may show that forward accessibility is equivalent to the following "rank condition", similar to (LCM3):

> **Rank Condition for the Scalar CM(F) Model**
>
> (CM2) For each initial condition $x_0^0 \in \mathbb{R}$ there exists $k \in \mathbb{Z}_+$ and a sequence $(u_1^0, \dots, u_k^0) \in O_w^k$ such that the derivative
>
> $$\Big[\frac{\partial}{\partial u_1} F_k\left(x_0^0, u_1^0, \dots, u_k^0\right) \mid \cdots \mid \frac{\partial}{\partial u_k} F_k\left(x_0^0, u_1^0, \dots, u_k^0\right)\Big] \qquad (7.4)$$
>
> is non-zero.

In the scalar linear case the control system (7.1) has the form

$$x_k = F x_{k-1} + G u_k$$

with F and G scalars. In this special case the derivative in (CM2) becomes exactly $[F^{k-1}G|\dots|FG|G]$, which shows that the rank condition (CM2) is a generalization of the controllability condition (LCM3) for the linear state space model. This connection will be strengthened when we consider multidimensional nonlinear models below.

Theorem 7.1.1 *The control model CM(F) is forward accessible if and only if the rank condition (CM2) is satisfied.*

A proof of this result would take us too far from the purpose of this book. It is similar to that of Proposition 7.1.2, and details may be found in [173, 174].

7.1.2 Continuous components for the scalar nonlinear model

Using the characterization of forward accessibility given in Theorem 7.1.1 we now show how this condition on CM(F) leads to the existence of a continuous component for the associated SNSS(F) model.

To do this we need to increase the strength of our assumptions on the noise process, as we did for the linear model or the random walk.

Density for the SNSS(F) Model

(SNSS3) The distribution Γ of W is absolutely continuous, with a density γ_w on $\mathbb{R}$ which is lower semicontinuous.

The control set for the SNSS(F) model is the open set

$$O_w := \{x \in \mathbb{R} : \gamma_w(x) > 0\}.$$

We know from the definitions that, with probability one, $W_k \in O_w$ for all $k \in \mathbb{Z}_+$. Commonly assumed noise distributions satisfying this assumption include those which possess a continuous density, such as the Gaussian model, or uniform distributions on bounded open intervals in $\mathbb{R}$.

We can now develop an explicit continuous component for such scalar nonlinear state space models.

Proposition 7.1.2 *Suppose that for the SNSS(F) model, the noise distribution satisfies (SNSS3), and that the associated control system CM(F) is forward accessible. Then the SNSS(F) model is a T-chain.*

PROOF Since CM(F) is forward accessible we have from Theorem 7.1.1 that the rank condition (CM2) holds. For simplicity of notation, assume that the derivative with respect to the kth disturbance variable is non-zero:

$$\frac{\partial F_k}{\partial w_k}(x_0^0, w_1^0, \dots, w_k^0) \neq 0 \tag{7.5}$$

with $(w_1^0, \dots, w_k^0) \in O_w^k$. Define the function $F^k : \mathbb{R} \times O_w^k \to \mathbb{R} \times O_w^{k-1} \times \mathbb{R}$ as

$$F^k(x_0, w_1, \dots, w_k) = (x_0, w_1, \dots, w_{k-1}, x_k)^\top$$

where $x_k = F_k(x_0, w_1, \dots, w_k)$. The total derivative of F^k can be computed as

$$DF^k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ \frac{\partial F_k}{\partial x_0} & \frac{\partial F_k}{\partial w_1} & \cdots & \frac{\partial F_k}{\partial w_k} \end{bmatrix}$$

which is evidently full rank at $(x_0^0, w_1^0, \dots, w_k^0)$. It follows from the Inverse Function Theorem that there exists an open set

$$B = B_{x_0^0} \times B_{w_1^0} \times \cdots \times B_{w_k^0}$$

containing $(x_0^0, w_1^0, \dots, w_k^0)$, and a smooth function $G^k : \{F^k\{B\}\} \to \mathbb{R}^{k+1}$ such that

$$G^k(F^k(x_0, w_1, \dots, w_k)) = (x_0, w_1, \dots, w_k)$$

for all $(x_0, w_1, \ldots, w_k) \in B$.

Taking G_k to be the final component of G^k, we see that for all $(x_0, w_1, \ldots, w_k) \in B$,

$$G_k(x_0, w_1, \ldots, w_{k-1}, x_k) = G_k(x_0, w_1, \ldots, w_{k-1}, F_k(x_0, w_1, \ldots, w_k)) = w_k.$$

We now make a change of variables, similar to the linear case. For any $x_0 \in B_{x_0^0}$, and any positive function $f\colon \mathbb{R} \to \mathbb{R}_+$,

$$\begin{aligned} P^k f\,(x_0) &= \int \cdots \int f(F_k(x_0, w_1, \ldots, w_k))\gamma_w(w_k) \cdots \gamma_w(w_1)\, dw_1 \ldots dw_k \\ &\geq \int_{B_{w_1^0}} \cdots \int_{B_{w_k^0}} f(F_k(x_0, w_1, \ldots, w_k))\gamma_w(w_k) \cdots \gamma_w(w_1)\, dw_1 \ldots dw_k. \end{aligned} \tag{7.6}$$

We will first integrate over w_k, keeping the remaining variables fixed. By making the change of variables

$$x_k = F_k(x_0, w_1, \ldots, w_k), \qquad w_k = G_k(x_0, w_1, \ldots, w_{k-1}, x_k)$$

so that

$$dw_k = |\frac{\partial G_k}{\partial x_k}(x_0, w_1, \ldots, w_{k-1}, x_k)|\, dx_k$$

we obtain for $(x_0, w_1, \ldots, w_{k-1}) \in B_{x_0^0} \times \ldots \times B_{w_{k-1}^0}$,

$$\int_{B_{w_k^0}} f(F_k(x_0, w_1, \ldots, w_k))\gamma_w(w_k)\, dw_k = \int_{\mathbb{R}} f(x_k) q_k(x_0, w_1, \ldots, w_{k-1}, x_k)\, dx_k \tag{7.7}$$

where we define, with $\xi := (x_0, w_1, \ldots, w_{k-1}, x_k)$,

$$q_k(\xi) := \mathbb{1}\{G^k(\xi) \in B\}\gamma_w(G_k(\xi))|\frac{\partial G_k}{\partial x_k}(\xi)|.$$

Since q_k is positive and lower semicontinuous on the open set $F^k\{B\}$, and zero on $F^k\{B\}^c$, it follows that q_k is lower semi-continuous on $\mathbb{R}^{k+1}$.

Define the kernel T_0 for an arbitrary bounded function f as

$$T_0 f\,(x_0) := \int \cdots \int f(x_k)\, q_k(\xi)\, \gamma_w(w_1) \cdots \gamma_w(w_{k-1})\, dw_1 \ldots dw_{k-1} dx_k. \tag{7.8}$$

The kernel T_0 is non-trivial at x_0^0 since

$$q_k(\xi^0)\gamma_w(w_1^0) \cdots \gamma_w(w_{k-1}^0) = |\frac{\partial G_k}{\partial x_k}(\xi^0)|\gamma_w(w_k^0)\gamma_w(w_1^0) \cdots \gamma_w(w_{k-1}^0) > 0,$$

where $\xi^0 = (x_0^0, w_1^0, \ldots, w_{k-1}^0, x_k^0)$. We will show that $T_0 f$ is lower semicontinuous on $\mathbb{R}$ whenever f is positive and bounded.

Since $q_k(x_0, w_1, \ldots, w_{k-1}, x_k)\gamma_w(w_1) \cdots \gamma_w(w_{k-1})$ is a lower semicontinuous function of its arguments in $\mathbb{R}^{k+1}$, there exists a sequence of positive, continuous functions $r_i\colon \mathbb{R}^{k+1} \to \mathbb{R}_+$, $i \in \mathbb{Z}_+$, such that for each i, the function r_i has bounded support and, as $i \uparrow \infty$,

$$r_i(x_0, w_1, \ldots, w_{k-1}, x_k) \uparrow q_k(x_0, w_1, \ldots, w_{k-1}, x_k)\gamma_w(w_1) \cdots \gamma_w(w_{k-1})$$

for each $(x_0, w_1, \ldots, w_{k-1}, x_k) \in \mathbb{R}^{k+1}$. Define the kernel T_i using r_i as

$$T_i f(x_0) := \int_{\mathbb{R}^k} f(x_k) r_i(x_0, w_1, \ldots, w_{k-1}, x_k)\, dw_1 \ldots dw_{k-1}\, dx_k.$$

It follows from the dominated convergence theorem that $T_i f$ is continuous for any bounded function f. If f is also positive, then as $i \uparrow \infty$,

$$T_i f(x_0) \uparrow T_0 f(x_0), \qquad x_0 \in \mathbb{R}$$

which implies that $T_0 f$ is lower semicontinuous when f is positive.

Using (7.6) and (7.7) we see that T_0 is a continuous component of P^k which is non-zero at x_0^0. From Theorem 6.2.4, the model is a T-chain as claimed. □

7.1.3 Simple bilinear model

The forward accessibility of the SNSS(F) model is usually immediate since the rank condition (CM2) is easily checked.

To illustrate the use of Proposition 7.1.2, and in particular the computation of the "controllability vector" (7.4) in (CM2), we consider the scalar example where $\boldsymbol{\Phi}$ is the bilinear state space model on $\mathsf{X} = \mathbb{R}$ defined in (SBL1) by

$$X_{k+1} = \theta X_k + b W_{k+1} X_k + W_{k+1}$$

where $\mathbf{W}$ is a disturbance process. To place this bilinear model into the framework of this chapter we assume

Density for the Simple Bilinear Model

(SBL2) The sequence $\mathbf{W}$ is a disturbance process on $\mathbb{R}$, whose marginal distribution Γ possesses a finite second moment, and a density γ_w which is lower semicontinuous.

Under (SBL1) and (SBL2), the bilinear model $\mathbf{X}$ is an SNSS(F) model with F defined in (2.7).

First observe that the one-step transition kernel P for this model cannot possess an everywhere non-trivial continuous component. This may be seen from the fact that $P(-1/b, \{-\theta/b\}) = 1$, yet $P(x, \{-\theta/b\}) = 0$ for all $x \neq -1/b$. It follows that the only positive lower semicontinuous function which is majorized by $P(\,\cdot\,, \{-\theta/b\})$ is zero, and thus any continuous component T of P must be trivial at $-1/b$: that is, $T(-1/b, \mathbb{R}) = 0$.

This could be anticipated by looking at the controllability vector (7.4). The first order controllability vector is

$$\frac{\partial F}{\partial u}(x_0, u_1) = bx_0 + 1,$$

which is zero at $x_0 = -1/b$, and thus the first order test for forward accessibility fails. Hence we must take $k \geq 2$ in (7.4) if we hope to construct a continuous component.

When $k = 2$ the vector (7.4) can be computed using the chain rule to give

$$\begin{aligned}
&\Big[\frac{\partial F}{\partial x}(x_1, u_2)\frac{\partial F}{\partial u}(x_0, u_1) \mid \frac{\partial F}{\partial u}(x_1, u_2)\Big] \\
&= \quad [(\theta + bu_2)(bx_0 + 1) \mid bx_1 + 1] \\
&= \quad [(\theta + bu_2)(bx_0 + 1) \mid \theta bx_0 + b^2 u_1 x_0 + bu_1 + 1]
\end{aligned}$$

which is non-zero for almost every $\binom{u_1}{u_2} \in \mathbb{R}^2$. Hence the associated control model is forward accessible, and this together with Proposition 7.1.2 gives

Proposition 7.1.3 *If (SBL1) and (SBL2) hold then the bilinear model is a T-chain.*

7.1.4 Multidimensional models

Most nonlinear processes that are encountered in applications cannot be modeled by a scalar Markovian model such as the SNSS(F) model. The more general NSS(F) model is defined by (NSS1), and we now analyze this in a similar way to the scalar model.

We again call the associated control system CM(F) with trajectories

$$x_k = F_k(x_0, u_1, \dots, u_k), \qquad k \in \mathbb{Z}_+, \tag{7.9}$$

forward accessible if the set of attainable states $A_+(x)$, defined as

$$A_+(x) := \bigcup_{k=0}^{\infty} \Big\{F_k(x, u_1, \dots, u_k) : u_i \in O_w, 1 \leq i \leq k\Big\}, \qquad k \geq 1, \tag{7.10}$$

has non-empty interior for every initial condition $x \in \mathsf{X}$.

To verify forward accessibility we define a further generalization of the controllability matrix introduced in (LCM3).

For $x_0 \in \mathsf{X}$ and a sequence $\{u_k : u_k \in O_w,\, k \in \mathbb{Z}_+\}$ let $\{\Xi_k,\, \Lambda_k : k \in \mathbb{Z}_+\}$ denote the matrices

$$\begin{aligned}
\Xi_{k+1} = \Xi_{k+1}(x_0, u_1, \dots, u_{k+1}) \quad &:= \quad \Big[\frac{\partial F}{\partial x}\Big]_{(x_k, u_{k+1})} \\
\Lambda_{k+1} = \Lambda_{k+1}(x_0, u_1, \dots, u_{k+1}) \quad &:= \quad \Big[\frac{\partial F}{\partial u}\Big]_{(x_k, u_{k+1})},
\end{aligned}$$

where $x_k = F_k(x_0, u_1 \cdots u_k)$. Let $C^k_{x_0} = C^k_{x_0}(u_1, \dots, u_k)$ denote the *generalized controllability matrix* (along the sequence $u_1, \dots, u_k$)

$$C^k_{x_0} := [\Xi_k \cdots \Xi_2 \, \Lambda_1 \mid \Xi_k \cdots \Xi_3 \, \Lambda_2 \mid \ \cdots \ \mid \Xi_k \, \Lambda_{k-1} \mid \Lambda_k] \,. \tag{7.11}$$

If F takes the linear form

$$F(x,u) = Fx + Gu \tag{7.12}$$

then the generalized controllability matrix again becomes

$$C^k_{x_0} = [F^{k-1}G \mid \cdots \mid G],$$

which is the controllability matrix introduced in (LCM3).

Rank Condition for the Multidimensional CM(F) Model

(CM3) For each initial condition $x_0 \in \mathbb{R}^n$, there exists $k \in \mathbb{Z}_+$ and a sequence $\vec{u}^0 = (u^0_1, \dots, u^0_k) \in O^k_w$ such that

$$\operatorname{rank} C^k_{x_0}(\vec{u}^0) = n. \tag{7.13}$$

The controllability matrix C^k_y is the derivative of the state $x_k = F(y, u_1, \dots, u_k)$ at time k with respect to the input sequence $(u_k^\top, \dots, u_1^\top)$. The following result is a consequence of this fact together with the Implicit Function Theorem and Sard's Theorem (see [107, 174] and the proof of Proposition 7.1.2 for details).

Proposition 7.1.4 *The nonlinear control model CM(F) satisfying (7.9) is forward accessible if and only the rank condition (CM3) holds.* □

To connect forward accessibility to the stochastic model (NSS1) we again assume that the distribution of W possesses a density.

Density for the NSS(F) Model

(NSS3) The distribution Γ of W possesses a density γ_w on $\mathbb{R}^p$ which is lower semicontinuous, and the control set for the NSS(F) model is the open set

$$O_w := \{x \in \mathbb{R} : \gamma_w(x) > 0\}.$$

Using an argument which is similar to, but more complicated than the proof of Proposition 7.1.2, we may obtain the following consequence of forward accessibility.

Proposition 7.1.5 *If the NSS(F) model satisfies the density assumption (NSS3), and the associated control model is forward accessible, then the state space* X *may be written as the union of open small sets, and hence the NSS(F) model is a T-chain.*

□

Note that this only guarantees the T-chain property: we now move on to consider the equally needed irreducibility properties of the NNS(F) models.

7.2 Minimal Sets and Irreducibility

We now develop a more detailed description of reachable states and topological irreducibility for the nonlinear state space NSS(F) model, and exhibit more of the interplay between the stochastic and topological communication structures for NSS(F) models.

Since one of the major goals here is to exhibit further the links between the behavior of the associated deterministic control model and the NSS(F) model, it is first helpful to study the structure of the accessible sets for the control system CM(F) with trajectories (7.9).

A large part of this analysis deals with a class of sets called minimal sets for the control system CM(F). In this section we will develop criteria for their existence and properties of their topological structure. This will allow us to decompose the state space of the corresponding NSS(F) model into disjoint, closed, absorbing sets which are both ψ-irreducible and topologically irreducible.

7.2.1 Minimality for the Deterministic Control Model

We define $A_+(E)$ to be the set of all states attainable by CM(F) from the set E at some time $k \geq 0$, and we let E^0 denote those states which cannot reach the set E:

$$A_+(E) := \bigcup_{x \in E} A_+(x) \qquad E^0 := \{x \in \mathsf{X} : A_+(x) \cap E = \emptyset\}.$$

Because the functions $F_k(\,\cdot\,, u_1, \dots, u_k)$ have the semi-group property

$$F_{k+j}(x_0, u_1, \dots, u_{k+j}) = F_j(F_k(x_0, u_1, \dots, u_k), u_{k+1}, \dots, u_{k+j}),$$

for $x_0 \in \mathsf{X}$, $u_i \in O_w$, $k, j \in \mathbb{Z}_+$, the set maps $\{A_+^k : k \in \mathbb{Z}_+\}$ also have this property: that is,

$$A_+^{k+j}(E) = A_+^k(A_+^j(E)), \qquad E \subset \mathsf{X}, \quad k, j \in \mathbb{Z}_+.$$

If $E \subset \mathsf{X}$ has the property that

$$A_+(E) \subset E$$

then E is called *invariant*. For example, for all $C \subset \mathsf{X}$, the sets $A_+(C)$ and C^0 are invariant, and since the closure, union, and intersection of invariant sets is invariant, the set

$$\Omega_+(C) := \bigcap_{N=1}^{\infty} \left\{ \overline{\bigcup_{k=N}^{\infty} A_+^k(C)} \right\} \tag{7.14}$$

is also invariant.

The following result summarizes these observations:

Proposition 7.2.1 *For the control system (7.9) we have for any $C \subset \mathsf{X}$,*

(i) *$A_+(C)$ and $\overline{A_+(C)}$ are invariant;*

(ii) *$\Omega_+(C)$ is invariant;*

(iii) *C^0 is invariant, and C^0 is also closed if the set C is open.* □

As a consequence of the assumption that the map F is smooth, and hence continuous, we then have immediately

Proposition 7.2.2 *If the associated CM(F) model is forward accessible then for the NSS(F) model:*

(i) *A closed subset $A \subset \mathsf{X}$ is absorbing for NSS(F) if and only if it is invariant for CM(F);*

(ii) *If $U \subset \mathsf{X}$ is open then for each $k \geq 1$ and $x \in \mathsf{X}$,*

$$A_+^k(x) \cap U \neq \emptyset \iff P^k(x, U) > 0;$$

(iii) *If $U \subset \mathsf{X}$ is open then for each $x \in \mathsf{X}$,*

$$A_+(x) \cap U \neq \emptyset \iff K_{a_\varepsilon}(x, U) > 0.$$

□

We now introduce minimal sets for the general CM(F) model.

Minimal sets

We call a set *minimal* for the deterministic control model CM(F) if it is (topologically) closed, invariant, and does not contain any closed invariant set as a proper subset.

For example, consider the LCM(F,G) model introduced in (1.4). The assumption (LCM2) simply states that the control set O_w is equal to $\mathbb{R}^p$.

In this case the system possesses a unique minimal set M which is equal to X_0, the range space of the controllability matrix, as described after Proposition 4.4.3. If the eigenvalue condition (LSS5) holds then this is the only minimal set for the LCM(F,G) model.

The following characterizations of minimality follow directly from the definitions, and the fact that both $\overline{A_+(x)}$ and $\Omega_+(x)$ are closed and invariant.

Proposition 7.2.3 *The following are equivalent for a nonempty set $M \subset \mathsf{X}$:*

(i) *M is minimal for CM(F);*

(ii) *$\overline{A_+(x)} = M$ for all $x \in M$;*

(iii) *$\Omega_+(x) = M$ for all $x \in M$.* □

7.2.2 M-Irreducibility and ψ-irreducibility

Proposition 7.2.3 asserts that any state in a minimal set can be "almost reached" from any other state. This property is similar in flavor to topological irreducibility for a Markov chain. The link between these concepts is given in the following central result for the NSS(F) model.

Theorem 7.2.4 *Let $M \subset \mathsf{X}$ be a minimal set for CM(F). If CM(F) is forward accessible and the disturbance process of the associated NSS(F) model satisfies the density condition (NSS3), then*

(i) *the set M is absorbing for NSS(F);*

(ii) *the NSS(F) model restricted to M is an open set irreducible (and so ψ-irreducible) T-chain.*

PROOF That M is absorbing follows directly from Proposition 7.2.3, proving $M = \overline{A_+(x)}$ for some x; Proposition 7.2.1, proving $\overline{A_+(x)}$ is invariant; and Proposition 7.2.2, proving any closed invariant set is absorbing for the NSS(F) model.

To see that the process restricted to M is topologically irreducible, let $x_0 \in M$, and let $U \subseteq \mathsf{X}$ be an open set for which $U \cap M \neq \emptyset$. By Proposition 7.2.3 we have $A_+(x_0) \cap U \neq \emptyset$. Hence by Proposition 7.2.2 $K_{a_\varepsilon}(x_0, U) > 0$, which establishes open set irreducibility. The process is then ψ-irreducible from Proposition 6.2.2 since we know it is a T-chain from Proposition 7.1.5. □

Clearly, under the conditions of Theorem 7.2.4, if X itself is minimal then the NSS(F) model is both ψ-irreducible and open set irreducible. The condition that X be minimal is a strong requirement which we now weaken by introducing a different form of "controllability" for the control system CM(F).

We say that the deterministic control system CM(F) is *indecomposable* if its state space X does not contain two disjoint closed invariant sets. This condition is clearly necessary for CM(F) to possess a unique minimal set. Indecomposability is not sufficient to ensure the existence of a minimal set: take $\mathsf{X} = \mathbb{R}$, $O_w = (0, 1)$, and

$$x_{k+1} = F(x_k, u_{k+1}) = x_k + u_{k+1},$$

so that all proper closed invariant sets are of the form $[t, \infty)$ for some $t \in \mathbb{R}$. This system is indecomposable, yet no minimal sets exist.

> Irreducible control models
>
> If CM(F) is indecomposable and also possesses a minimal set M, then CM(F) will be called *M-irreducible.*

If CM(F) is M-irreducible it follows that $M^0 = \emptyset$: otherwise M and M^0 would be disjoint nonempty closed invariant sets, contradicting indecomposability. To establish necessary and sufficient conditions for M-irreducibility we introduce a concept from dynamical systems theory. A state $x^\star \in \mathsf{X}$ is called *globally attracting* if for all $y \in \mathsf{X}$,

$$x^\star \in \Omega_+(y).$$

The following result easily follows from the definitions.

Proposition 7.2.5 **(i)** *The nonlinear control system (7.9) is M-irreducible if and only if a globally attracting state exists.*

(ii) *If a globally attracting state $x^\star$ exists then the unique minimal set is equal to $\overline{A_+(x^\star)} = \Omega_+(x^\star)$.* □

We can now provide the desired connection between irreducibility of the nonlinear control system and ψ-irreducibility for the corresponding Markov chain.

Theorem 7.2.6 *Suppose that CM(F) is forward accessible and the disturbance process of the associated NSS(F) model satisfies the density condition (NSS3).*

Then the NSS(F) model is ψ-irreducible if and only if CM(F) is M-irreducible.

Proof If the NSS(F) model is ψ-irreducible, let $x^\star$ be any state in $\operatorname{supp}\psi$, and let U be any open set containing $x^\star$. By definition we have $\psi(U) > 0$, which implies that $K_{a_\varepsilon}(x, U) > 0$ for all $x \in \mathsf{X}$. By Proposition 7.2.2 it follows that $x^\star$ is globally attracting, and hence CM(F) is M-irreducible by Proposition 7.2.5.

Conversely, suppose that CM(F) possesses a globally attracting state, and let U be an open petite set containing $x^\star$. Then $A_+(x) \cap U \neq \emptyset$ for all $x \in \mathsf{X}$, which by Proposition 7.2.2 and Proposition 5.5.4 implies that the NSS(F) model is ψ-irreducible for some ψ. □

7.3 Periodicity for nonlinear state space models

We now look at the periodic structure of the nonlinear NSS(F) model to see how the cycles of Section 5.4.3 can be further described, and in particular their topological structure elucidated.

We first demonstrate that minimal sets for the deterministic control model CM(F) exhibit periodic behavior. This periodicity extends to the stochastic framework in a natural way, and under mild conditions on the deterministic control system, we will see that the period is in fact trivial, so that the chain is aperiodic.

7.3.1 Periodicity for control models

To develop a periodic structure for CM(F) we mimic the construction of a cycle for an irreducible Markov chain. To do this we first require a deterministic analogue of small sets: we say that the set C is *k-accessible* from the set B, for any $k \in \mathbb{Z}_+$, if for each $y \in B$,

$$C \subset A_+^k(y).$$

This will be denoted $B \xrightarrow{k} C$. From the Implicit Function Theorem, in a manner similar to the proof of Proposition 7.1.2, we can immediately connect k-accessibility with forward accessibility.

Proposition 7.3.1 *Suppose that the CM(F) model is forward accessible. Then for each $x \in \mathsf{X}$, there exist open sets B_x, $C_x \subset \mathsf{X}$, with $x \in B_x$ and an integer $k_x \in \mathbb{Z}_+$ such that $B_x \xrightarrow{k_x} C_x$.* □

In order to construct a cycle for an irreducible Markov chain, we first constructed a ν_n-small set A with $\nu_n(A) > 0$. A similar construction is necessary for CM(F).

Lemma 7.3.2 *Suppose that the CM(F) model is forward accessible. If M is minimal for CM(F) then there exists an open set $E \subset M$, and an integer $n \in \mathbb{Z}_+$, such that $E \xrightarrow{n} E$.*

PROOF Using Proposition 7.3.1 we find that there exist open sets B and C, and an integer k with $B \xrightarrow{k} C$, such that $B \cap M \neq \emptyset$. Since M is invariant, it follows that

$$C \subset A_+(B \cap M) \subset M, \tag{7.15}$$

and by Proposition 7.2.1, minimality, and the hypothesis that the set B is open,

$$A_+(x) \cap B \neq \emptyset \tag{7.16}$$

for every $x \in M$.

Combining (7.15) and (7.16) it follows that $A_+^m(c) \cap B \neq \emptyset$ for some $m \in \mathbb{Z}_+$, and $c \in C$. By continuity of the function F we conclude that there exists an open set $E \subset C$ such that

$$A_+^m(x) \cap B \neq \emptyset \qquad \text{for all} \;\; x \in E.$$

The set E satisfies the conditions of the lemma with $n = m+k$ since by the semi-group property,

$$A_+^n(x) = A_+^k(A_+^m(x)) \supset A_+^k(A_+^m(x) \cap B) \supset C \supset E$$

for all $x \in E$ □

Call a finite ordered collection of disjoint closed sets $\mathbf{G} := \{G_i : 1 \leq i \leq d\}$ a *periodic orbit* if for each i,

$$A_+^1(G_i) \subset G_{i+1} \qquad i = 1, \ldots, d \quad (\text{mod } d)$$

The integer d is called the *period* of $\mathbf{G}$.

The cyclic result for CM(F) is given in

Theorem 7.3.3 *Suppose that the function $F{:}\,\mathsf{X} \times O_w \to \mathsf{X}$ is smooth, and that the system CM(F) is forward accessible.*

If M is a minimal set, then there exists an integer $d \geq 1$, and disjoint closed sets $\mathbf{G} = \{G_i : 1 \leq i \leq d\}$ such that $M = \bigcup_{i=1}^d G_i$, and $\mathbf{G}$ is a periodic orbit. It is unique in the sense that if $\mathbf{H}$ is another periodic orbit whose union is equal to M with period d', then d' divides d, and for each i the set H_i may be written as a union of sets from $\mathbf{G}$.

PROOF Using Lemma 7.3.2 we can fix an open set E with $E \subset M$, and an integer k such that $E \xrightarrow{k} E$. Define $I \subset \mathbb{Z}_+$ by

$$I := \{n \geq 1 : E \xrightarrow{n} E\} \tag{7.17}$$

The semi-group property implies that the set I is closed under addition: for if $i, j \in I$, then for all $x \in E$,

$$A_+^{i+j}(x) = A_+^i(A_+^j(x)) \supset A_+^j(E) \supset E.$$

Let d denote g.c.d.(I). The integer d will be called the *period* of M, and M will be called *aperiodic* when $d = 1$.

For $1 \leq i \leq d$ we define

$$G_i := \{x \in M : \bigcup_{k=1}^{\infty} A_+^{kd-i}(x) \cap E \neq \emptyset\}. \tag{7.18}$$

By Proposition 7.2.1 it follows that $M = \bigcup_{i=1}^d G_i$.

Since E is an open subset of M, it follows that for each $i \in \mathbb{Z}_+$, the set G_i is open in the relative topology on M. Once we have shown that the sets $\{G_i\}$ are disjoint, it will follow that they are closed in the relative topology on M. Since M itself is closed, this will imply that for each i, the set G_i is closed.

We now show that the sets $\{G_i\}$ are disjoint. Suppose that on the contrary $x \in G_i \cap G_j$ for some $i \neq j$. Then there exists $k_i,\ k_j \in \mathbb{Z}_+$ such that

$$A_+^{k_i d-i}(y) \cap E \neq \emptyset \qquad \text{and} \qquad A_+^{k_j d-j}(y) \cap E \neq \emptyset \tag{7.19}$$

when $y = x$. Since E is open, we may find an open set $O \subset \mathsf{X}$ containing x such that (7.19) holds for all $y \in O$.

By Proposition 7.2.1, there exists $v \in E$ and $n \in \mathbb{Z}_+$ such that

$$A_+^n(v) \cap O \neq \emptyset. \tag{7.20}$$

By (7.20), (7.19), and since $E \xrightarrow{k_0} E$ we have for $\delta = i, j$, and all $z \in E$,

$$\begin{aligned} A_+^{k_0+k_\delta d-\delta+n+k_0}(z) &\supset A_+^{k_0+k_\delta d-\delta+n}(E) \\ &\supset A_+^{k_0+k_\delta d-\delta}(A_+^n(v) \cap O) \\ &\supset A_+^{k_0}(A_+^{k_\delta d-\delta}(A_+^n(v) \cap O) \cap E) \supset E. \end{aligned}$$

This shows that

$$2k_0 + k_\delta d - \delta + n \in I$$

for $\delta = i,\ j$, and this contradicts the definition of d. We conclude that the sets $\{G_i\}$ are disjoint.

We now show that $\mathbf{G}$ is a periodic orbit. Let $x \in G_i$, and $u \in O_w$. Since the sets $\{G_i\}$ form a disjoint cover of M and since M is invariant, there exists a unique $1 \leq j \leq d$ such that $F(x, u) \in G_j$. It follows from the semi-group property that $x \in G_{j-1}$, and hence $i = j - 1$.

The uniqueness of this construction follows from the definition given in equation (7.18). □

The following consequence of Theorem 7.3.3 further illustrates the topological structure of minimal sets.

Proposition 7.3.4 *Under the conditions of Theorem 7.3.3, if the control set O_w is connected, then the periodic orbit* **G** *constructed in Theorem 7.3.3 is precisely equal to the connected components of the minimal set M.*

In particular, in this case M is aperiodic if and only if it is connected.

PROOF First suppose that M is aperiodic. Let $E \xrightarrow{n} E$, and consider a fixed state $v \in E$.

By aperiodicity and Lemma D.7.4 there exists an integer N_0 with the property that

$$e \in A_+^k(v) \tag{7.21}$$

for all $k \geq N_0$. Since $A_+^k(v)$ is the continuous image of the connected set $v \times O_w^k$, the set

$$A_+(A_+^{N_0}(v)) = \bigcup_{k=N_0}^{\infty} A_+^k(v) \tag{7.22}$$

is connected. Its closure is therefore also connected, and by Proposition 7.2.1 the closure of the set (7.22) is equal to M.

The periodic case is treated similarly. First we show that for some $N_0 \in \mathbb{Z}_+$ we have

$$G_d = \overline{\bigcup_{k=N_0}^{\infty} A_+^{kd}(v)},$$

where d is the period of M, and each of the sets $A_+^{kd}(v)$, $k \geq N_0$, contains v.

This shows that G_d is connected. Next, observe that

$$G_1 = \overline{A_+^1(G_d)},$$

and since the control set O_w and G_d are both connected, it follows that G_1 is also connected. By induction, each of the sets $\{G_i : 1 \leq i \leq d\}$ is connected. □

7.3.2 Periodicity

All of the results described above dealing with periodicity of minimal sets were posed in a purely deterministic framework. We now return to the stochastic model described by (NSS1)-(NSS3) to see how the deterministic formulation of periodicity relates to the stochastic definition which was introduced for Markov chains in Section 5.4.

As one might hope, the connections are very strong.

Theorem 7.3.5 *If the NSS(F) model satisfies Conditions (NSS1)-(NSS3) and the associated control model CM(F) is forward accessible then:*

(i) *If M is a minimal set, then the restriction of the NSS(F) model to M is a ψ-irreducible T-chain, and the periodic orbit $\{G_i : 1 \leq i \leq d\} \subset M$ whose existence is guaranteed by Theorem 7.3.3 is ψ-a.e. equal to the d-cycle constructed in Theorem 5.4.4;*

(ii) *If CM(F) is M-irreducible, and if its unique minimal set M is aperiodic, then the NSS(F) model is a ψ-irreducible aperiodic T-chain.*

PROOF The proof of (i) follows directly from the definitions, and the observation that by reducing E if necessary, we may assume that the set E which is used in the proof of Theorem 7.3.3 is small. Hence the set E plays the same role as the small set used in the proof of Theorem 5.2.1. The proof of (ii) follows from (i) and Theorem 7.2.4. □

7.4 Forward Accessible Examples

We now see how specific models may be viewed in this general context. It will become apparent that without making any unnatural assumptions, both simple models such as the dependent parameter bilinear model, and relatively more complex nonlinear models such as the gumleaf attractor with noise and adaptive control models can be handled within this framework.

7.4.1 The dependent parameter bilinear model

The dependent parameter bilinear model is a simple NSS(F) model where the function F is given in (2.14) by

$$F\left(\binom{Y}{\theta}, \binom{Z}{W}\right) = \begin{pmatrix} \alpha\theta + Z \\ \theta Y + W \end{pmatrix} \tag{7.23}$$

Using Proposition 7.1.4 it is easy to see that the associated control model is forward accessible, and then the model is easily analyzed. We have

Proposition 7.4.1 *The dependent parameter bilinear model* $\boldsymbol{\Phi}$ *satisfying Assumptions (DBL1)–(DBL2) is a T-chain. If further there exists some one* $z^* \in O_z$ *such that*

$$\left|\frac{z^*}{1-\alpha}\right| < 1, \tag{7.24}$$

then $\boldsymbol{\Phi}$ *is* ψ*-irreducible and aperiodic .*

PROOF With the noise $\binom{\mathbf{Z}}{\mathbf{W}}$ considered a "control", the first order controllability matrix may be computed to give

$$C^1_{\theta,y} = \frac{\partial\binom{\theta_1}{Y_1}}{\partial\binom{Z_1}{W_1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The control model is thus forward accessible, and hence $\boldsymbol{\Phi} = \binom{\boldsymbol{\theta}}{\mathbf{Y}}$ is a T-chain.

Suppose now that the bound (7.24) holds for z^* and let w^* denote any element of $O_w \subseteq \mathbb{R}$. If Z_k and W_k are set equal to z^* and w^* respectively in (7.23) then as $k \to \infty$

$$\begin{pmatrix} \theta_k \\ Y_k \end{pmatrix} \to x^* := \begin{pmatrix} z^*(1-\alpha)^{-1} \\ w^*(1-\alpha)(1-\alpha-z^*)^{-1} \end{pmatrix}$$

The state x^* is globally attracting, and it immediately follows from Proposition 7.2.5 and Theorem 7.2.6 that the chain is ψ-irreducible. Aperiodicity then follows from the fact that any cycle must contain the state x^*. □

7.4.2 The gumleaf attractor

Consider the NSS(F) model whose sample paths evolve to create the version of the "gumleaf attractor" illustrated in Figure 2.5. This model is given in (2.11) by

$$X_n = \begin{pmatrix} X_n^a \\ X_n^b \end{pmatrix} = \begin{pmatrix} -1/X_{n-1}^a + 1/X_{n-1}^b \\ X_{n-1}^a \end{pmatrix} + \begin{pmatrix} W_n \\ 0 \end{pmatrix}$$

which is of the form (NSS1), with the associated CM(F) model defined as

$$F\left(\binom{x^a}{x^b}, u\right) = \begin{pmatrix} -1/x^a + 1/x^b \\ x^a \end{pmatrix} + \begin{pmatrix} u \\ 0 \end{pmatrix}. \tag{7.25}$$

From the formulae

$$\frac{\partial F}{\partial x} = \begin{pmatrix} (1/x^a)^2 & -(1/x^b)^2 \\ 1 & 0 \end{pmatrix} \qquad \frac{\partial F}{\partial u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we see that the second order controllability matrix is given by

$$C_{x_0}^2(u_1, u_2) = \begin{bmatrix} (1/x_1^a)^2 & 1 \\ 1 & 0 \end{bmatrix}$$

where $x_0 = \binom{x_0^a}{x_0^b}$ and $x_1^a = -1/x_0^a + 1/x_0^b + u_1$. Hence, since $C_{x_0}^2$ is full rank for all x_0, u_1 and u_2, it follows that the control system is forward accessible. Applying Proposition 7.2.6 gives

Proposition 7.4.2 *The NSS(F) model (2.11) is a T-chain if the disturbance sequence* **W** *satisfies Condition (NSS3).*

7.4.3 The adaptive control model

The adaptive control model described by (2.21)-(2.23) is of the general form of the NSS(F) model and the results of the previous section are well suited to the analysis of this specific example

An apparent difficulty with this model is that the state space X is not an open subset of Euclidean space, so that the general results obtained for the NSS(F) model may not seem to apply directly. However, given our assumptions on the model, the interior of the state space, $(\sigma_z, \frac{\sigma_z}{1-\alpha^2}) \times \mathbb{R}^2$, is absorbing, and is reached in one step with probability one from each initial condition. Hence to obtain a continuous component, and to address periodicity for the adaptive model, we can apply the general results obtained for the nonlinear state space models by first restricting $\boldsymbol{\Phi}$ to the interior of X.

Proposition 7.4.3 *If (SAC1) and (SAC2) hold for the adaptive control model defined by (2.21-2.23), and if $\sigma_z^2 < 1$, then $\boldsymbol{\Phi}$ is a ψ-irreducible and aperiodic T-chain.*

PROOF To prove the result we show that the associated deterministic control model for the nonlinear state space model defined by (2.21-2.23) is forward accessible, and that for the associated deterministic control system, a globally attracting point exists.

The second-order controllability matrix has the form

$$C^2_{\Phi_0}(Z_2,W_2,Z_1,W_1) := \frac{\partial(\Sigma_2,\tilde\theta_2,Y_2)^\top}{\partial(Z_2,W_2,Z_1,W_1)^\top} = \begin{bmatrix} 0 & \frac{-2\alpha^2\sigma_w^2\Sigma_1^2Y_1}{(\Sigma_1Y_1^2+\sigma_w^2)^2} & 0 & 0 \\ \bullet & \bullet & 1 & \bullet \\ \bullet & \bullet & 0 & 1 \end{bmatrix}$$

where "$\bullet$" denotes a variable which does not affect the rank of the controllability matrix. It is evident that $C^2_{\Phi_0}$ is full rank whenever $Y_1 = \tilde\theta_0 Y_0 + W_1$ is non-zero. This shows that for each initial condition $\Phi_0 \in \mathsf{X}$, the matrix $C^2_{\Phi_0}$ is full rank for a.e. $\{(Z_1,W_1),(Z_2,W_2)\} \in \mathbb{R}^4$, and so the associated control model is forward accessible, and hence the stochastic model is a T-chain by Proposition 7.1.5.

It is easily checked that if $\binom{\mathbf{Z}}{\mathbf{W}}$ is set equal to zero in (2.21)-(2.22) then, since $\alpha < 1$ and $\sigma_z^2 < 1$,

$$\Phi_k \to (\frac{\sigma_z^2}{1-\alpha^2},0,0)^\top \qquad \text{as } k \to \infty.$$

This shows that the control model associated with the Markov chain $\boldsymbol{\Phi}$ is M-irreducible, and hence by Proposition 7.2.6 the chain itself is ψ-irreducible. The limit above also shows that every element of a cycle $\{G_i\}$ for the unique minimal set must contain the point $(\frac{\sigma_z^2}{1-\alpha^2},0,0)$. From Proposition 7.3.4 it follows that the chain is aperiodic. □

7.5 Equicontinuity and the nonlinear state space model

7.5.1 e-Chain properties of nonlinear state space models

We have seen in this chapter that the NSS(F) model is a T-chain if the noise variable, viewed as a control, can "steer the state process $\boldsymbol{\Phi}$" to a sufficiently large set of states.

If the forward accessibility property does not hold then the chain must be analyzed using different methods. The process is always a Feller Markov chain, because of the continuity of F, as shown in Proposition 6.1.2. In this section we search for conditions under which the process $\boldsymbol{\Phi}$ is also an e-chain.

To do this we consider the *derivative process* associated with the NSS(F) model, defined by $\Delta_0 = I$ and

$$\Delta_{k+1} = [DF\,(\Phi_k, w_{k+1})]\Delta_k, \qquad k \in \mathbb{Z}_+ \tag{7.26}$$

where $\boldsymbol{\Delta}$ takes values in the set of $n \times n$-matrices, and DF denotes the derivative of F with respect to its first variable.

Since $\Delta_0 = I$ it follows from the chain rule and induction that the derivative process is in fact the derivative of the present state with respect to the initial state: that is,

$$\Delta_k = \frac{d}{d\Phi_0}\Phi_k \qquad \text{for all } k \in \mathbb{Z}_+.$$

The main result in this section connects stability of the derivative process with equicontinuity of the transition function for $\boldsymbol{\Phi}$. Since the system (7.26) is closely related to the system (NSS1), linearized about the sample path $(\Phi_0, \Phi_1, \ldots)$, it is reasonable to expect that the stability of $\boldsymbol{\Phi}$ will be closely related to the stability of $\boldsymbol{\Delta}$.

Theorem 7.5.1 *Suppose that (NSS1)-(NSS3) hold for the NSS(F) model. Then letting Δ_k denote the derivative of Φ_k with respect to Φ_0, $k \in \mathbb{Z}_+$, we have*

(i) *if for some open convex set $N \subset \mathsf{X}$,*

$$\mathsf{E}[\sup_{\Phi_0 \in N} \|\Delta_k\|] < \infty \tag{7.27}$$

then for all $x \in N$,

$$\frac{d}{dx}\mathsf{E}_x[\Phi_k] = \mathsf{E}_x[\Delta_k];$$

(ii) *suppose that (7.27) holds for all sufficiently small neighborhoods N of each $y_0 \in \mathsf{X}$, and further that for any compact set $C \subset \mathsf{X}$,*

$$\sup_{y \in C} \sup_{k \geq 0} \mathsf{E}_y[\|\Delta_k\|] < \infty.$$

Then $\boldsymbol{\Phi}$ is an e-chain.

PROOF The first result is a consequence of the Dominated Convergence Theorem.

To prove the second result, let $f \in \mathcal{C}_c(\mathsf{X}) \cap \mathcal{C}^\infty(\mathsf{X})$. Then

$$\Big|\frac{d}{dx}P^k f\,(x)\Big| = \Big|\frac{d}{dx}\mathsf{E}_x[f(\Phi_k)]\Big| \leq \|f'\|_\infty \mathsf{E}_x[\|\Delta_k\|]$$

which by the assumptions of (ii), implies that the sequence of functions $\{P^k f : k \in \mathbb{Z}_+\}$ is equicontinuous on compact subsets of X. Since $C^\infty \cap C_c$ is dense in C_c, this completes the proof. □

It may seem that the technical assumption (7.27) will be difficult to verify in practice. However, we can immediately identify one large class of examples by considering the case where the i.i.d. process $\mathbf{W}$ is uniformly bounded. It follows from the smoothness condition on F that $\sup_{\Phi_0 \in N} \|\Delta_k\|$ is almost surely finite for any compact subset $N \subset \mathsf{X}$, which shows that in this case (7.27) is trivially satisfied.

The following result provides another large class of models for which (7.27) is satisfied. Observe that the conditions imposed on $\mathbf{W}$ in Proposition 7.5.2 are satisfied for any i.i.d. Gaussian process. The proof is straightforward.

Proposition 7.5.2 *For the Markov chain defined by (NSS1)-(NSS3), suppose that F is a rational function of its arguments, and that for some $\varepsilon_0 > 0$,*

$$\mathsf{E}[\exp(\varepsilon_0 |W_1|)] < \infty.$$

Then letting Δ_k denote the derivative of Φ_k with respect to Φ_0, we have for any compact set $C \subset \mathsf{X}$, and any $k \geq 0$,

$$\mathsf{E}[\sup_{\Phi_0 \in C} \|\Delta_k\|] < \infty.$$

Hence under these conditions,

$$\frac{d}{dx}\mathsf{E}_x[\Phi_k] = \mathsf{E}_x[\Delta_k].$$

□

7.5.2 Linear state space models

We can easily specialize Theorem 7.5.1 to give conditions under which a linear model is an e-chain.

Proposition 7.5.3 *Suppose the LSS(F,G) model* **X** *satisfies (LSS1) and (LSS2), and that the eigenvalue condition (LSS5) also holds. Then* $\boldsymbol{\Phi}$ *is an e-chain.*

PROOF Using the identity $X_m = F^m X_0 + \sum_{i=0}^{m-1} F^i G W_{m-i}$ we see that

$$\Delta_k = F^m,$$

which tends to zero exponentially fast, by Lemma 6.3.4. The conditions of Theorem 7.5.1 are thus satisfied, which completes the proof. □

Observe that Proposition 7.5.3 uses the eigenvalue condition (LSS5), the same assumption which was used in Proposition 4.4.3 to obtain ψ-irreducibility for the Gaussian model, and the same condition that will be used to obtain stability in later chapters.

The analogous Proposition 6.3.3 uses controllability to give conditions under which the linear state space model is a T-chain. Note that controllability is not required here.

Other specific nonlinear models, such as bilinear models, can be analyzed similarly using this approach.

7.6 Commentary

We have already noted that in the degenerate case where the control set O_w consists of a single point, the NSS(F) model defines a semi-dynamical system with state space X, and in fact many of the concepts introduced in this chapter are generalizations of standard concepts from dynamical systems theory.

Three standard approaches to the qualitative theory of dynamical systems are *topologial dynamics* whose principal tool is point set topology; *ergodic theory*, where one assumes (or proves, frequently using a compactness argument) the existence of an ergodic invariant measure; and finally, the *direct method of Lyapunov*, which concerns criteria for stability.

The latter two approaches will be developed in a stochastic setting in Parts II and III. This chapter essentially focused on generalizations of the first approach, which is also based upon, to a large extent, the structure and existence of minimal sets. Two excellent expositions in a purely deterministic and control-free setting are the books by Bhatia and Szegö [22] and Brown [37]. Saperstone [234] considers infinite dimensional spaces so that, in particular, the methods may be applied directly to the dynamical system on the space of probability measures which is generated by a Markov processes.

The connections between control theory and irreducibility described here are taken from Meyn [169] and Meyn and Caines [174, 173]. The dissertations of Chan [41] and Mokkadem [187], and also Diebolt and Guégan [64], treat discrete time nonlinear state space models and their associated control models. Diebolt in [63] considers nonlinear models with additive noise of the form $\Phi_{k+1} = F(\Phi_k) + W_{k+1}$ using an approach which is very different to that described here.

Jakubsczyk and Sontag in [107] present a survey of the results obtainable for forward accessible discrete time control systems in a purely deterministic setting. They give a different characterization of forward accessibility, based upon the rank of an associated Lie algebra, rather than a controllability matrix.

The origin of the approach taken in this chapter lies in the often cited paper by Stroock and Varadahn [260]. There it is shown that the support of the distribution of a diffusion process may be characterized by considering an associated control model. Ichihara and Kunita in [101] and Kliemann in [138] use this approach to develop an ergodic theory for diffusions. The *invariant control sets* of [138] may be compared to minimal sets as defined here.

At this stage, introduction of the e-chain class of models is not well-motivated. The reader who wishes to explore them immediately should move to Chapter 12.

In Duflo [69], a condition closely related to the stability condition which we impose on $\boldsymbol{\Delta}$ is used to obtain the Central Limit Theorem for a nonlinear state space model. Duflo assumes that the function F satisfies

$$|F(x,w) - F(y,w)| \leq \alpha(w)|x-y|$$

where α is a function on O_w satisfying, for some sufficiently large m,

$$\mathsf{E}[\alpha(W)^m] < 1.$$

It is easy to see that any process $\boldsymbol{\Phi}$ generated by a nonlinear state space model satisfying this bound is an e-chain.

For models more complex than the linear model of Section 7.5.2 it will not be as easy to prove that $\boldsymbol{\Delta}$ converges to zero, so a lengthier stability analysis of this derivative process may be necessary. Since $\boldsymbol{\Delta}$ is essentially generated by a random linear system it is therefore likely to either converge to zero or evanesce.

It seems probable that the stochastic Lyapunov function approach of Kushner [149] or Khas'minskii [134], or a more direct analysis based upon limit theorems for products of random matrices as developed in, for instance, Furstenberg and Kesten [84] will be well suited for assessing the stability of $\boldsymbol{\Delta}$.