19

Generalized Classification Criteria

We have now developed a number of simple criteria, solely involving the one step transition function, which enable us to classify quite general Markov chains. We have seen, for example, that the equivalences in Theorem 11.0.1, Theorem 13.0.1, or Theorem 15.0.1 give an effective approach to the analysis of many systems.

For more complex models, however, the analysis of the simple one step drift

$$\Delta V(x) = \int P(x, dy)[V(y) - V(x)]$$

towards petite sets may not be straightforward, or indeed may even be impracticable. Even though we know from the powerful converse theory in the theorems just mentioned that for most forms of stability, there must be at least one V with the one step drift ΔV suitably negative, finding such a function may well be non-trivial.

In this chapter we conclude our approach to stochastic stability by giving a number of more general drift criteria which enable the classification of chains where the one-step criteria are not always straightforward to construct. All of these variations are within the general framework described previously. The steps to be used in practice are, we hope, clear from the preceding chapters, and follow the route reiterated in Appendix A.

There are three generalizations of the drift criteria which we consider here.

(a) *State dependent drift conditions,* which allow for negative drifts after a number of steps $n(x)$ depending on the state x from which the chain starts;

(b) *Path or history dependent drift conditions,* which allow for functions of the whole past of the process to show a negative drift;

(c) *Mixed or "average" drift conditions,* which allow for functions whose drift varies in direction, but which is negative in a suitably "averaged" way.

For each of these we also indicate the application of the method by example. The state-dependent drift technique is used to analyze random walk on $\mathbb{R}^2_+$ and a model of invasion/defense, where simple one-step drift conditions seem almost impossible to construct; the history-dependent methods are shown to be suited to bilinear models with random coefficients, where again one-step drift conditions seem to fail; and, finally, the mixed drift analysis gives us a criterion for ladder processes, and in particular the Markovian representation of the full GI/G/1 queue, to be ergodic.

19.1 State-dependent drifts

19.1.1 The state-dependent drift criteria

In this section we consider consequences of state-dependent drift conditions of the form

$$\int P^{n(x)}(x,dy)V(y) \quad \le \quad g[V(x),n(x)], \qquad x \in C^c, \tag{19.1}$$

where $n(x)$ is a function from X to $\mathbb{Z}_+$, g is a function depending on which type of stability we seek to establish, and C is an appropriate petite set.

The function $n(x)$ here provides the state-dependence of the drift conditions, since from any x we must wait $n(x)$ steps for the drift to be negative.

In order to develop results in this framework we work with an "embedded" chain $\hat{\boldsymbol{\Phi}}$. Using $n(x)$ we define the new transition law $\{\hat{P}(x,A)\}$ by

$$\hat{P}(x,A) = P^{n(x)}(x,A), \qquad x \in \mathsf{X},\ A \in \mathcal{B}(\mathsf{X}), \tag{19.2}$$

and let $\hat{\boldsymbol{\Phi}}$ be the corresponding Markov chain. This Markov chain may be constructed explicitly as follows. The time $n(x)$ is a (trivial) stopping time. Let $s(k)$ denote its iterates: that is, along any sample path, $s(0)=0$, $s(1)=n(x)$ and

$$s(k+1) = s(k) + n(\Phi_{s(k)}).$$

Then it follows from the strong Markov property that

$$\hat{\Phi}_k = \Phi_{s(k)}, \qquad k \ge 0 \tag{19.3}$$

is a Markov chain with transition law $\hat{P}$.

Let $\hat{\mathcal{F}}_k = \mathcal{F}_{s(k)}$ be the σ-field generated by the events "before $s(k)$": that is,

$$\hat{\mathcal{F}}_k := \{A : A \cap \{s(k) \le n\} \in \mathcal{F}_n, n \ge 0\}.$$

We let $\hat{\tau}_A, \hat{\sigma}_A$ denote the first return and first entry index to A respectively for the chain $\hat{\boldsymbol{\Phi}}$. Clearly $s(k)$ and the events $\{\hat{\sigma}_A \ge k\}$, $\{\hat{\tau}_A \ge k\}$ are $\hat{\mathcal{F}}_{k-1}$-measurable for any $A \in \mathcal{B}(\mathsf{X})$.

Note that $s(\hat{\tau}_C)$ denotes the time of first return to C by the original chain $\boldsymbol{\Phi}$ along an embedded path, defined by

$$s(\hat{\tau}_C) := \sum_{0}^{\hat{\tau}_C - 1} n(\hat{\Phi}_k). \tag{19.4}$$

From (19.3) we have

$$s(\hat{\tau}_C) \ge \tau_C, \qquad s(\hat{\sigma}_C) \ge \sigma_C, \qquad \text{a.s. } [\mathsf{P}_*]. \tag{19.5}$$

These relations will enable us to use the drift equations (19.1), with which we will bound the index at which $\hat{\boldsymbol{\Phi}}$ reaches C, to bound the hitting times on C by the original chain.

We first give a state-dependent criterion for Harris recurrence.

Theorem 19.1.1 *Suppose that $\boldsymbol{\Phi}$ is a ψ-irreducible chain on X, and let $n(x)$ be a function from X to $\mathbb{Z}_+$. The chain is Harris recurrent if there exists a non-negative function V unbounded off petite sets, and some petite set C satisfying*

$$\int P^{n(x)}(x,dy)V(y) \quad \le \quad V(x), \qquad x \in C^c. \tag{19.6}$$

PROOF The proof is an adaptation of the proof of Theorem 9.4.1.

Let $C_0 = C$, and let $C_n = \{x \in \mathsf{X} : V(x) \le n\}$. By assumption, the sets C_n, $n \in \mathbb{Z}_+$, are petite.

Now suppose by way of contradiction that $\boldsymbol{\Phi}$ is not Harris recurrent. By Theorem 8.0.1 the chain is either recurrent, but not Harris recurrent, or the chain is transient. In either case, we show that there exists an initial condition x_0 such that

$$\mathsf{P}_{x_0}\{(\boldsymbol{\Phi} \in C \text{ i.o.})^c \cap (V(\Phi_k) \to \infty)\} > 0. \tag{19.7}$$

Firstly, if the chain is transient, then by Theorem 8.3.5 each C_n is uniformly transient, and hence $V(\Phi_k) \to \infty$ as $k \to \infty$ a.s. $[\mathsf{P}_*]$, and so (19.7) holds.

Secondly, if $\boldsymbol{\Phi}$ is recurrent, then the state space may be written as

$$\mathsf{X} = H \cup N \tag{19.8}$$

where $H = N^c$ is a maximal Harris set and $\psi(N) = 0$; this follows from Theorem 9.0.1. Since for each n the set C_n is petite we have $C_n \rightsquigarrow H$, and hence by Theorem 9.1.3,

$$\{\boldsymbol{\Phi} \in C_n \text{ i.o.}\} \subset \{\boldsymbol{\Phi} \in H \text{ i.o.}\} \qquad \text{a.s. } [\mathsf{P}_*].$$

It follows that the inclusion $\{\liminf V(\Phi_n) < \infty\} \subset \{\boldsymbol{\Phi} \in H \text{ i.o.}\}$ holds with probability one. Thus (19.7) holds for any $x_0 \in N$, and if the chain is not Harris, we know N is non-empty.

Now from (19.7) there exists $M \in \mathbb{Z}_+$ with

$$\mathsf{P}_{x_0}\{(\Phi_k \in C^c, k \ge M) \cap (V(\Phi_k) \to \infty)\} > 0 :$$

letting $\mu = P^M(x_0, \cdot\,)$, we have by conditioning at time M,

$$\mathsf{P}_{\mu}\{(\sigma_C = \infty) \cap (V(\Phi_k) \to \infty)\} > 0. \tag{19.9}$$

We now show that (19.9) leads to a contradiction when (19.6) holds.

Define the chain $\hat{\boldsymbol{\Phi}}$ as in (19.3). We can write (19.6), for every k, as

$$\mathsf{E}[V(\hat{\Phi}_{k+1}) \mid \hat{\mathcal{F}}_k] \le V(\hat{\Phi}_k) \qquad \text{a.s. } [\mathsf{P}_*]$$

when $\hat{\sigma}_C > k$, $k \in \mathbb{Z}_+$.

Let $M_i = V(\hat{\Phi}_i)\mathbb{1}\{\hat{\sigma}_C \ge i\}$. Using the fact that $\{\hat{\sigma}_C \ge k\} \in \hat{\mathcal{F}}_{k-1}$, we have that

$$\mathsf{E}[M_k \mid \hat{\mathcal{F}}_{k-1}] = \mathbb{1}\{\hat{\sigma}_C \ge k\}\mathsf{E}[V(\hat{\Phi}_k) \mid \hat{\mathcal{F}}_{k-1}] \le \mathbb{1}\{\hat{\sigma}_C \ge k\}V(\hat{\Phi}_{k-1}) \le M_{k-1}.$$

Hence $(M_k, \hat{\mathcal{F}}_k)$ is a positive supermartingale, so that from Theorem D.6.2 there exists an almost surely finite random variable M_∞ such that $M_k \to M_\infty$ a.s. as $k \to \infty$. From the construction of M_i, either $\hat{\sigma}_C < \infty$ in which case $M_\infty = 0$, or $\hat{\sigma}_C = \infty$ in which case $\limsup_{k\to\infty} V(\hat{\Phi}_k) = M_\infty < \infty$ a.s.

Since $\sigma_C < \infty$ whenever $\hat{\sigma}_C < \infty$, this shows that for any initial distribution μ,

$$\mathsf{P}_{\mu}\{\{\sigma_C < \infty\} \cup \{\liminf_{n\to\infty} V(\Phi_n) < \infty\}^c\} = 1.$$

This contradicts (19.9), and hence the chain is Harris recurrent. □

We next prove a state-dependent criterion for positive recurrence.

Theorem 19.1.2 *Suppose that $\boldsymbol{\Phi}$ is a ψ-irreducible chain on X, and let $n(x)$ be a function from X to $\mathbb{Z}_+$. The chain is positive Harris recurrent if there exists some petite set C, a non-negative function V bounded on C, and a positive constant b satisfying*

$$\int P^{n(x)}(x,dy)V(y) \quad \leq \quad V(x) - n(x) + b\mathbb{1}_C(x), \qquad x \in \mathsf{X} \tag{19.10}$$

in which case for all x

$$\mathsf{E}_x[\tau_C] \leq V(x) + b. \tag{19.11}$$

PROOF The state-dependent drift criterion for positive recurrence is a direct consequence of the f-ergodicity results of Theorem 14.2.2, which tell us that without any irreducibility or other conditions on $\boldsymbol{\Phi}$, if f is a non-negative function and

$$\int P(x,dy)V(y) \quad \leq \quad V(x) - f(x) + b\mathbb{1}_C(x), \qquad x \in \mathsf{X} \tag{19.12}$$

for some set C then for all $x \in \mathsf{X}$

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_C - 1} f(\Phi_k)\Big] \leq V(x) + b. \tag{19.13}$$

Again define the chain $\hat{\boldsymbol{\Phi}}$ as in (19.3). From (19.10) we can use (19.13) for $\hat{\boldsymbol{\Phi}}$, with $f(x)$ taken as $n(x)$, to deduce that

$$\mathsf{E}_x\Big[\sum_{k=0}^{\hat{\tau}_C - 1} n(\hat{\Phi}_k)\Big] \leq V(x) + b. \tag{19.14}$$

But we have by adding the lengths of the embedded times $n(x)$ along any sample path that from (19.4)

$$\sum_{k=0}^{\hat{\tau}_C - 1} n(\hat{\Phi}_k) = s(\hat{\tau}_C) \geq \tau_C.$$

Thus from (19.14) and the fact that V is bounded on the petite set C, we have that $\boldsymbol{\Phi}$ is positive Harris using the one-step criterion in Theorem 13.0.1, and the bound (19.11) follows also from (19.14). □

We conclude the section with a state-dependent criterion for geometric ergodicity.

Theorem 19.1.3 *Suppose that $\boldsymbol{\Phi}$ is a ψ-irreducible chain on X, and let $n(x)$ be a function from X to $\mathbb{Z}_+$. The chain is geometrically ergodic if it is aperiodic and there exists some petite set C, a non-negative function $V \geq 1$ and bounded on C, and positive constants $\lambda < 1$ and $b < \infty$ satisfying*

$$\int P^{n(x)}(x,dy)V(y) \quad \leq \quad \lambda^{n(x)}[V(x) + b\mathbb{1}_C(x)]. \tag{19.15}$$

When (19.15) holds,

$$\sum_n r^n \|P^n(x,\,\cdot\,) - \pi\| \leq RV(x), \qquad x \in \mathsf{X} \tag{19.16}$$

for some constants $R < \infty$ and $r > 1$.

PROOF Suppose that (19.15) holds, and define

$$V'(x) = 2(V(x) - 1/2) \geq 1.$$

Then we can write (19.15) as

$$\begin{aligned} \int \hat{P}(x,dy)V'(y) &\leq \lambda^{n(x)}[2V(x) + 2b\mathbb{1}_C(x)] - 1 \\ &= \lambda^{n(x)}[V'(x) + 1 + 2b\mathbb{1}_C(x)] - 1 \end{aligned} \tag{19.17}$$

Without loss of generality we will therefore assume that V itself satisfies the inequality

$$\int \hat{P}(x,dy)V(y) \leq \lambda^{n(x)}[V(x) + 1 + b\mathbb{1}_C(x)] - 1. \tag{19.18}$$

We now adapt the proof of Theorem 15.2.5. Define the random variables

$$Z_k = \kappa^{s(k)} V(\hat{\Phi}_k)$$

for $k \in \mathbb{Z}_+$. It follows from (19.18) that for $\kappa = \lambda^{-1}$, since $\kappa^{s(k+1)}$ is $\hat{\mathcal{F}}_k$-measurable,

$$\begin{aligned} \mathsf{E}[Z_{k+1} \mid \hat{\mathcal{F}}_k] &= \kappa^{s(k+1)} \mathsf{E}[V(\hat{\Phi}_{k+1}) \mid \hat{\mathcal{F}}_k] \\ &\leq \kappa^{s(k+1)} \{\kappa^{-n(\Phi_k)}[V(\Phi_k) + 1 + b\mathbb{1}_C(\Phi_k)] - 1\} \\ &= Z_k - \kappa^{s(k+1)} + \kappa^{s(k)} + \kappa^{s(k)} b\mathbb{1}_C(\Phi_k). \end{aligned}$$

Using Proposition 11.3.2 we have

$$\mathsf{E}_x\Big[\sum_{k=0}^{\hat{\tau}_C - 1} [\kappa^{s(k+1)} - \kappa^{s(k)}]\Big] \leq Z_0(x) + \mathsf{E}_x\Big[\sum_{k=0}^{\hat{\tau}_C - 1} \kappa^{s(k)} b\mathbb{1}_C(\hat{\Phi}_k)\Big].$$

Collapsing the sum on the left and using the fact that only the first term in the sum on the right is non-zero, we get

$$\mathsf{E}_x[\kappa^{s(\hat{\tau}_C)} - 1] \leq V(x) + b\mathbb{1}_C(x). \tag{19.19}$$

Since $V < \infty$ and V is assumed bounded on C, and again using the fact that $s(\hat{\tau}_C) > \tau_C$, we have from Theorem 15.0.1 (ii) that the chain is geometrically ergodic.

The final bound in (19.16) comes from the fact that for some r, an upper bound on the state-dependent constant term in (19.16) is shown in Theorem 15.4.1 to be given by

$$R(x) = \mathsf{E}_x[\kappa^{\tau_C}] \leq \mathsf{E}_x[\kappa^{s(\hat{\tau}_C)}] \leq (2+b)V(x)$$

since $V \geq 1$. □

19.1.2 Models on $\mathbb{R}^2_+$

State dependent criteria appear to be of most use in analyzing multidimensional models, especially those on the positive orthant of Euclidean space. This is because, although the normal one-step drift conditions may work in the interior of such spaces, the constraints on the faces of the orthant can imply that drift is not negative in this part of the space.

We illustrate this in a simple case when the space is $\mathbb{R}^2_+ = \{(x,y), x \geq 0, y \geq 0\}$.

Consider the case of random walk restricted to the positive orthant. Let $Z_k = (Z_k(1), Z_k(2))$ be a sequence of i.i.d. random variables in $\mathbb{R}^2$ and define the chain $\boldsymbol{\Phi}$ by

$$(\Phi_n(1), \Phi_n(2)) = ([\Phi_{n-1}(1) + Z_n(1)]^+, [\Phi_{n-1}(2) + Z_n(2)]^+). \tag{19.20}$$

Let us assume that for each coordinate we have negative increments: that is,

$$\mathsf{E}[Z_k(1)] < 0, \qquad \mathsf{E}[Z_k(2)] < 0.$$

This assumption ensures that the chain is a $\delta_{(0,0)}$-irreducible chain with all compact sets petite. To see this note that there exists $h > 0$ such that

$$\mathsf{P}(Z_k(1) < -h) > h, \qquad \mathsf{P}(Z_k(2) < -h) > h,$$

and so for any square $S_w = \{x \leq w, y \leq w\}$ we have that, choosing $m \geq w/h$

$$P^m((x,y),(0,0)) > h^{2m} > 0, \qquad (x,y) \in S_w.$$

This provides $\delta_{(0,0)}$-irreducibility, and moreover shows that S_w is small, with $\nu = \delta_{0,0}$ in (5.14).

We will also assume that the second moments of the increments are finite:

$$\mathsf{E}[Z_k^2(1)] < \infty, \qquad \mathsf{E}[Z_k^2(2)] < \infty.$$

Thus it follows from Proposition 14.4.1 that each of the marginal random walks on $[0,\infty)$ is positive Harris with stationary measures π_1, π_2 satisfying

$$\beta_1 := \int z\pi_1(dz) < \infty, \qquad \beta_2 := \int z\pi_2(dz) < \infty. \tag{19.21}$$

Of course, from this we could establish positivity merely by noting that $\pi = \pi_1 \times \pi_2$ is invariant for the bivariate chain. However, in order to illustrate the methods of this section we will establish that $\boldsymbol{\Phi}$ is positive Harris by considering the test function $V(x,y) = x + y$: this also gives us a bound on the hitting times of rectangles that the more indirect result does not provide.

By choosing M large enough we can ensure that the truncated versions of the increments are also negative, so that for some $\varepsilon > 0$

$$\mathsf{E}[Z_k(1)\mathbb{1}\{Z_k(1) \geq -M\}] < -\varepsilon, \qquad \mathsf{E}[Z_k(2)\mathbb{1}\{Z_k(2) \geq -M\}] < -\varepsilon.$$

This ensures that on the set $A(M) = \{x \geq M, y \geq M\}$, we have that (19.10) holds with $n(x,y) = 1$ in the usual manner.

Now consider the strip $A_1(M,m) = \{x \leq M, y \geq m\}$, and fix $(x,y) \in A_1(M,m)$.

Let us choose a given fixed number of steps n, and choose $m > (M+1)n$. At each step in the time period $\{0,\ldots,n\}$ the expected value of $\Phi_n(2)$ decreases in expectation by at least ε. Moreover, from (19.21) and the f-norm ergodic result (14.5) we have that by convergence there is a constant c_0 such that for all n

$$\mathsf{E}_{(0,y)}[\Phi_n(1)] \leq c_0 \tag{19.22}$$

independent of y. From stochastic monotonicity we also have that for all $x \le M$, if τ_0 denotes the first hitting time on $\{0\}$ for the marginal chain $\Phi_n(1)$

$$\begin{array}{rcl} \mathsf{E}_{(x,y)}[\Phi_n(1)\mathbb{1}\{\tau_0 > n\}] & \le & \mathsf{E}_{(M,y)}[\Phi_n(1)\mathbb{1}\{\tau_0 > n\}] \\ & := & \zeta_M(n) \end{array} \tag{19.23}$$

which is finite and tends to zero as $n \to \infty$, from Theorem 14.2.7, independent of y. Let us choose n large enough that $\zeta_M(n) \le \varepsilon_0$.

We thus have from the Markov property

$$\begin{array}{rcl} \mathsf{E}_{(x,y)}[\Phi_n(1) + \Phi_n(2)] & = & \mathsf{E}_{(x,y)}[\Phi_n(2)] + \mathsf{E}_{(x,y)}[\Phi_n(1)\mathbb{1}\{\tau_0 > n\}] \\ & & + \mathsf{E}_{(x,y)}[\Phi_n(1)\mathbb{1}\{\tau_0 \le n\}] \\ & \le & y - n\varepsilon + \varepsilon_0 + c_0. \end{array} \tag{19.24}$$

Thus for $x \le M$, we have uniform negative n-step drift in the region $A_1(M,m)$ provided

$$n\varepsilon > M + \varepsilon_0 + c_0$$

as required.

A similar construction enables us to find that for fixed large n the n-step drift in the region $A_2(m,M)$ is negative also. Thus we have shown

Theorem 19.1.4 *If the bivariate random walk on $\mathbb{R}^2_+$ has negative mean increments and finite second moments in both coordinates then it is positive Harris recurrent, and for sets $A(m) = \{x \ge m, y \ge m\}$ with m large, and some constant c,*

$$\mathsf{E}_{(x,y)}[\tau_{A(m)}] \le c(x+y). \tag{19.25}$$

In this example, we do not use the full power of the results of Section 19.1. Only three values of $n(x,y)$ are used, and indeed it is apparent from the construction in (19.24) that we could have treated the whole chain on the region

$$\{x \ge M+n\} \cup \{y \ge M+n\}$$

for the same n. In this case the n-skeleton $\{\Phi_{nk}\}$ would be shown to be positive recurrent, and it follows from the fact that the invariant measure for $\{\Phi_k\}$ is also invariant for $\{\Phi_{nk}\}$ that the original chain is positive Harris: see Chapter 10. This example does, however, indicate the steps that we could go through to analyze less homogeneous models, and also indicates that it is easier to analyze the boundaries or non-standard regions independently of the interior or standard region of the space without the need to put the results together for a single fixed skeleton.

19.1.3 An invasion/antibody model

We conclude this section with the analysis of an invasion/antibody model on a countable space, illustrating another type of model where control of state-dependent drift is useful.

Models for competition between two groups can be modeled as bivariate processes on the integer-valued quadrant $\mathbb{Z}^2_+ = \{i, j \in \mathbb{Z}_+\}$. Consider such a process in discrete time with the first coordinate process $\Phi_n(1)$ denoting the numbers of invaders and the second coordinate process $\Phi_n(2)$ denoting the numbers of defenders.

(A1) Suppose first that the defenders and invaders mutually tend to reduce the opposition numbers when both groups are present, even though "reinforcements" may join either side. Thus on the interior of the space, denoted $I = \{i, j \geq 1\}$, we assume that for some $\varepsilon_i, \varepsilon_j \geq \varepsilon > 1/2$

$$\mathsf{E}_{i,j}[\Phi_1(1) + \Phi_1(2)] \leq (i - \varepsilon_i) + (j - \varepsilon_j) \leq i + j - 2\varepsilon, \qquad i, j > 1. \tag{19.26}$$

Such behavior might model, for example, antibody action against invasive bodies where there is physical attachment of at least one antibody to each invader and then both die: in such a context we would have $\varepsilon_i = \varepsilon_j = 1$.

(A2) On one boundary, when the defender numbers reach the level 0, if the invaders are above a threshold level d the body dies in which case the invaders also die and the chain drops to $(0,0)$, so that

$$P((i,0),(0,0)) = 1, \qquad i > d; \tag{19.27}$$

otherwise a new population of antibodies or defenders of finite mean size is generated. These assumptions are of course somewhat unrealistic and clearly with more delicate arguments can be made much more general if required.

(A3) Much more critically, on the other boundary, when the invader numbers fall to level 0, and the defenders are of size $j > 0$, a new "invading army" is raised to bring the invaders to size N, where N is a random variable concentrated on $\{j+1, j+2, \ldots, j+d\}$ for the same threshold d, so that

$$\sum_{k=1}^{d} P((0,j),(j+k,j) = 1 : \tag{19.28}$$

this distribution being concentrated above j represents the physically realistic concept that a new invasion will fail instantly if the invading population is not at least the size of the defending population. The bounded size of the increment is purely for convenience of exposition.

Note that the chain is $\delta_{(0,0)}$-irreducible under the assumptions A1-A3, regardless of the behavior at zero. Thus the model can be formulated to allow for a stationary distribution at $(0,0)$ (i.e extinction) or for rebirth and a more generally distributed stationary distribution over the whole of $\mathbb{Z}_2^+$. The only restriction we place in general is that the increments from $(0,0)$ have finite mean: here we will not make this more explicit as it does not affect our analysis.

Let us, to avoid unrewarding complexities, add to (19.26) the additional condition that the model is "left-continuous": that is, has bounded negative increments defined by

$$P((i,j),(i-l,j-k) = 0, \qquad i,j > 0, \qquad k,l > 1 : \tag{19.29}$$

this would be appropriate if the chain were embedded at the jumps of a continuous time process, for example.

To evaluate positive recurrence of the model, we use the test function $V(i,j) = [i+j]/\beta$, where $\beta < \varepsilon$ is to be chosen.

Analysis of this model in the interior of the space is not difficult: by using (V2) with $V(i,j)$ on $I = \{i, j \geq 1\}$, we have that $\mathsf{E}_{i,j}[\tau_{I^c}] < (i+j)/\beta$ from the assumption

(A1). The difficulty with such multidimensional models is that even though they reach I^c in a finite mean time, they may then "escape" along one or both of the boundaries. It is in this region that the tools of Section 19.1 are useful in assisting with the classification of the model.

Starting at $B_1(c) = \{(i,0), i > c\}$, the infinite boundary edge above c, we have that the value of $V(\Phi_1)$ is zero if $c > d$, so that (19.10) also holds with $n = 1$ provided we choose $c > \max(d, \beta^{-1})$.

On the other infinite boundary edge, denoted $B_2(c) = \{(0,j), j > c\}$, however, we have positive one step drift of the function V. Now from the starting point $(0,j)$, let us consider the $(j+1)$-step drift. This is bounded above by $[j + d - 2j\varepsilon]/\beta$ and so we have (19.10) also holds with $n(j) = j + 1$ provided

$$[j + d - 2j\varepsilon]/\beta < -j - 1,$$

which will hold provided $\beta < 2\varepsilon - 1$ and we then choose $c > (d + \beta)/(2\varepsilon - 1 - \beta)$.

Consequently we can assert that, writing $C = I \cup B_2(c) \cup B_1(c)$ with c satisfying both these constraints, the mean time

$$\mathsf{E}_{(i,j)}[\tau_C] \le [i + j]/\beta$$

regardless of the threshold level d, and so the invading strategy is successful in overcoming the antibody defense.

Note that in this model there is no fixed time at which the drift from all points on the boundary $B_2(c)$ is uniformly negative, no matter what the value of c chosen. Thus, state-dependent drift conditions appear needed to analyze this model.

To test for geometric ergodicity we use the function $V(i,j) = \exp(\alpha i) + \exp(\alpha j)$ and adopt the approach in Section 16.3.

We assume that the increments in the model have uniformly geometrically decreasing tails and bounded second moments: specifically, we assume each coordinate process satisfies, for some $\gamma > 0$,

$$\begin{aligned} \theta_i(\gamma) &:= \textstyle\sum_{k \ge i-1} \exp(\gamma k) \mathsf{P}_{i,j}(\Phi_1(1) = i + k) < \infty, \qquad j \ge 1 \\ \theta'_j(\gamma) &:= \textstyle\sum_{k \ge j-1} \exp(\gamma k) \mathsf{P}_{i,j}(\Phi_1(2) = j + k) < \infty, \qquad i \ge 1 \end{aligned} \tag{19.30}$$

and

$$\begin{aligned} \textstyle\sum_{k \ge i-1} k^2 \mathsf{P}_{i,j}(\Phi_1(1) = i + k) &< D_1, \qquad j \ge 1 \\ \textstyle\sum_{k \ge j-1} k^2 \mathsf{P}_{i,j}(\Phi_1(2) = j + k) &< D_2, \qquad i \ge 1. \end{aligned} \tag{19.31}$$

Then on the interior set I we have, for $\alpha < \gamma$

$$\begin{aligned} \textstyle\sum_j P((r,s),(i,j)) V(i,j) &\le \exp(\alpha r)[\theta_i(\alpha) - 1] \\ &\qquad + \exp(\alpha s)[\theta'_j(\alpha) - 1] \\ &\le \alpha \exp(\alpha r)(-\varepsilon_r/2) \\ &\qquad + \alpha \exp(\alpha s)(-\varepsilon_s/2) \end{aligned} \tag{19.32}$$

for small enough α, using a Taylor series expansion and the uniform conditions (19.30) and (19.31). Thus (19.15) holds with $n = 1$ and $\lambda = 1 - \alpha\varepsilon/2$.

Starting at $B_1(c)$, (19.15) also obviously holds provided we choose c large enough. On the other infinite boundary edge $B_2(c) = \{(0,j), j > c\}$ we have a similar construction for the $j+1$-step drift. We have, using the uniform bounds (19.31) assumed on the variances

$$\begin{aligned}\textstyle\sum_j P^{j+1}((0,s),(i,j))V(i,j) \quad &\leq \quad \exp(\alpha(j+d))[1-\varepsilon/2]^j \\ &\qquad + \exp(\alpha s)[1-\varepsilon/2]^j\end{aligned} \tag{19.33}$$

and so for α suitably small, we have (19.15) holding again as required. □

19.2 History-dependent drift criteria

The approach through Dynkin's Formula to obtaining bounds on hitting times of appropriate sets allows a straightforward generalization to more complex, history-dependent, test functions with very little extra effort above that expended already.

Rather than considering a fixed function V of the state Φ_k, we will now let $\{V_k : k \in \mathbb{Z}_+\}$ denote a family of non-negative Borel measurable functions $V_k\colon \mathsf{X}^{k+1} \to \mathbb{R}_+$. By imposing the appropriate "drift condition" on the stochastic process $\{V_k = V_k(\Phi_0, \dots, \Phi_k)\}$, we will obtain generalized criteria for stability and non-stability. The value of this generalization will be illustrated below in an application to an autoregressive model with random coefficients.

19.2.1 Generalized criteria for positivity and nullity

We first consider, in the time-varying context, drift conditions on such a family $\{V_k : k \in \mathbb{Z}_+\}$ for chains to be positive or to be null. We call a sequence $\{V_k, \mathcal{F}_k^\Phi\}$ *adapted* if V_k is measurable with respect to $\mathcal{F}_k^\Phi$ for each k.

The following condition generalizes (V2).

Generalized Negative Drift Condition

There exists a set $C \in \mathcal{B}(\mathsf{X})$, and an adapted sequence $\{V_k, \mathcal{F}_k^\Phi\}$ such that, for some $\varepsilon > 0$,

$$\mathsf{E}[V_{k+1} \mid \mathcal{F}_k^\Phi] \leq V_k - \varepsilon \qquad \text{a.s. } [\mathsf{P}_*] \tag{19.34}$$

when $\sigma_C > k$, $k \in \mathbb{Z}_+$.

As usual the condition that $\sigma_C > k$ means that $\Phi_i \in C^c$ for each i between 0 and k. Since C will usually be assumed "small" in some sense (either petite, or compact), (19.34) implies that there is a drift towards the "center" of the state space when $\boldsymbol{\Phi}$ is "large" in exactly the same way that (V2) does.

From these generalized drift conditions and Dynkin's Formula we find

Theorem 19.2.1 *If $\{V_k\}$ satisfies (19.34) then*

$$\mathsf{E}_x[\tau_C] \leq \begin{cases} \varepsilon^{-1} V_0(x) & x \in C^c \\ 1 + \varepsilon^{-1} P V_0\,(x) & x \in C \end{cases}$$

Hence if C is petite and $\sup_{x\in C} \mathsf{E}_x[V_0(\Phi_1)] < \infty$ then $\boldsymbol{\Phi}$ is regular.

Proof The proof follows immediately from Proposition 11.3.3 by letting $Z_k = V_k$, $\varepsilon_k = \varepsilon$, exactly as in Theorem 11.3.4. □

There is a similar generalization of the drift criterion for determining whether a given chain is null.

Generalized Positive Drift Condition

There exists a set $C \in \mathcal{B}(\mathsf{X})$, and an adapted sequence $\{V_k, \mathcal{F}_k^{\Phi}\}$ with

$$\mathsf{E}[V_{k+1} \mid \mathcal{F}_k^{\Phi}] \geq V_k \qquad \text{a.s. } [\mathsf{P}_*], \tag{19.35}$$

when $\sigma_C > k$, $k \in \mathbb{Z}_+$.

Clearly the process $V_k \equiv 1$ satisfies (19.35), so we will need some auxiliary conditions to prove anything specific when (19.35) holds.

Theorem 19.2.2 *Suppose that $\{V_k\}$ satisfies (19.35), and let $x_0 \in C^c$ be such that*

$$V_0(x_0) > V_k(x_0, \ldots, x_k), \qquad x_k \in C,\; k \in \mathbb{Z}_+. \tag{19.36}$$

Suppose moreover the conditional absolute increments have bounded means: that is, for some constant $B < \infty$,

$$\mathsf{E}[|V_k - V_{k-1}| \mid \mathcal{F}_{k-1}^{\Phi}] \leq B. \tag{19.37}$$

Then $\mathsf{E}_{x_0}[\tau_C] = \infty$.

Proof The proof of Theorem 11.5.1 goes through without change, although in this case the functions V_k in that proof are not taken simply as $V(\Phi_k)$ but as $V_k(\Phi_0, \ldots, \Phi_k)$. □

19.2.2 Generalized criteria for geometric ergodicity

We can extend the results of Chapter 15 in a similar way when the space admits a topology. In order to derive such criteria we need to adapt the sequence $\{V_k\}$ appropriately to the topology. Let us call the whole sequence $\{V_k\}$ norm-like if there exists a norm-like function $V : \mathsf{X} \to \mathbb{R}_+$ with the property

$$V_k(x_0, \dots, x_k) \geq V(x_k) \geq 0 \tag{19.38}$$

for all $k \in \mathbb{Z}_+$ and all $x_i \in \mathsf{X}$.

The criterion for such a family $\{V_k\}$ generalizes (15.35), which we showed in Lemma 15.2.8 to be equivalent to (V4).

Generalized Geometric Drift Condition

There exists $\lambda < 1$, $L < \infty$ and an adapted norm-like sequence $\{V_k, \mathcal{F}_k^{\Phi}\}$ such that

$$\mathsf{E}_x[V_{k+1} \mid \mathcal{F}_k^{\Phi}] \leq \lambda V_k + L \qquad \text{a.s. } [\mathsf{P}_*], \quad k \in \mathbb{Z}_+. \tag{19.39}$$

Theorem 19.2.3 *Suppose that $\boldsymbol{\Phi}$ is an irreducible aperiodic T-chain. If the generalized geometric drift condition (19.39) holds, and if V_0 is uniformly bounded on compact subsets of X, then there exists $R < \infty$ and $r > 1$ such that*

$$\sum_{n=1}^{\infty} r^n \| P^n(x, \cdot\,) - \pi \|_f \leq R(V_0\,(x) + 1), \qquad n \in \mathbb{Z}_+,\ x \in \mathsf{X}$$

where $f = V + 1$ and V is as defined in (19.38). In particular, $\boldsymbol{\Phi}$ is then f-geometrically ergodic.

Proof Let $\lambda < \rho < 1$, and define the precompact set C and the constant $\varepsilon > 0$ by

$$C = \{x \in \mathsf{X} : V(x) \leq \frac{2L}{\rho - \lambda} + 1\}, \qquad \varepsilon = \frac{\rho - \lambda}{2}.$$

Then for all $k \in \mathbb{Z}_+$,

$$\mathsf{E}[V_{k+1} \mid \mathcal{F}_k^{\Phi}] \leq \rho V_k + \Big\{[L + (\rho - \lambda)] - \frac{\rho - \lambda}{2}(V(\Phi_k) + 1)\Big\} - \frac{\rho - \lambda}{2}(V(\Phi_k) + 1)$$

Hence $\mathsf{E}[V_{k+1} \mid \mathcal{F}_k^{\Phi}] \leq \rho V_k - \varepsilon f(\Phi_k)$ when $\Phi_k \in C^c$. Letting $Z_k = r^k V_k$, where $r = \rho^{-1}$, we then have $\mathsf{E}[Z_k \mid \mathcal{F}_{k-1}^{\Phi}] - Z_{k-1} \leq -\varepsilon r^k f(\Phi_{k-1})$, when $\Phi_{k-1} \in C^c$. We now use Dynkin's formula to deduce that for all $x \in \mathsf{X}$,

$$\begin{aligned} 0 \le \mathsf{E}_x[z_{\tau_C^m}] &= \mathsf{E}_x[Z_1] + \mathsf{E}_x\Big[\Big(\sum_{k=2}^{\tau_C^m} \mathsf{E}[Z_k \mid \mathcal{F}_{k-1}^{\Phi}] - Z_{k-1}\Big)\mathbb{1}(\tau_C \ge 2)\Big] \\ &\le \mathsf{E}_x[Z_1] - \mathsf{E}_x\Big[\sum_{k=2}^{\tau_C^m} \varepsilon r^k f(\Phi_{k-1})\mathbb{1}(\tau_C \ge 2)\Big] \end{aligned}$$

This and the Monotone Convergence Theorem shows that for all $x \in \mathsf{X}$,

$$\mathsf{E}_x\Big[\sum_{k=1}^{\tau_C} r^k f(\Phi_{k-1})\Big] \le \varepsilon^{-1} r \mathsf{E}_x[V_1] + rV(x).$$

This completes the proof, since $\mathsf{E}_x[V_1] + V(x) \le \lambda V_0(x) + L + V_0(x)$ by (19.39) and (19.38). □

19.2.3 Generalized criteria for non-evanescence and transience

A general criterion for Harris recurrence on a topological space can be obtained from the following history dependent drift condition, which generalizes (V1).

> Generalized Non-positive Drift Condition
>
> There exists a compact set $C \subset \mathsf{X}$, and an adapted norm-like sequence $\{V_k, \mathcal{F}_k^{\Phi}\}$ such that
>
> $$\mathsf{E}[V_{k+1} \mid \mathcal{F}_k^{\Phi}] \le V_k \qquad \text{a.s. } [\mathsf{P}_*], \tag{19.40}$$
>
> when $\sigma_C > k$, $k \in \mathbb{Z}_+$.

Theorem 19.2.4 *If (19.40) holds then $\boldsymbol{\Phi}$ is non-evanescent. Hence if $\boldsymbol{\Phi}$ is a ψ-irreducible T-chain and (19.40) holds for a norm-like sequence and a compact C, then $\boldsymbol{\Phi}$ is Harris recurrent.*

PROOF The proof is almost identical to that of Theorem 9.4.1. If $\mathsf{P}_x\{\boldsymbol{\Phi} \to \infty\} > 0$ for some $x \in \mathsf{X}$, then (9.30) holds, so that for some M

$$\mathsf{P}_\mu\{\{\sigma_C = \infty\} \cap \{\boldsymbol{\Phi} \to \infty\}\} > 0, \tag{19.41}$$

where $\mu = P^M(x, \cdot)$.

This time let $M_i = V_i \mathbb{1}\{\sigma_C \ge i\}$. Again we have that $(M_k, \mathcal{F}_k^{\Phi})$ is a positive supermartingale, since

$$\mathsf{E}[M_k \mid \mathcal{F}_{k-1}^{\Phi}] = \mathbb{1}\{\sigma_C \ge k\}\mathsf{E}[V_k \mid \mathcal{F}_{k-1}^{\Phi}] \le \mathbb{1}\{\sigma_C \ge k\}V_{k-1} \le M_{k-1}.$$

Hence there exists an almost surely finite random variable M_∞ such that $M_k \to M_\infty$ as $k \to \infty$.

But as in Theorem 9.4.1, either $\sigma_C < \infty$ in which case $M_\infty = 0$, or $\sigma_C = \infty$ which contradicts (19.41). Hence $\boldsymbol{\Phi}$ is again non-evanescent.

The Harris recurrence when $\boldsymbol{\Phi}$ is a T-chain follows as usual by Theorem 9.2.2. □

Finally, we give a criterion for transience using a time-varying test function.

Generalized Non-negative Drift Condition

There exists a set $A \in \mathcal{B}(\mathsf{X})$, and a uniformly bounded, adapted sequence $\{V_k, \mathcal{F}_k^{\Phi}\}$ such that

$$\mathsf{E}[V_{k+1} \mid \mathcal{F}_k^{\Phi}] \geq V_k \qquad \text{a.s. } [\mathsf{P}_*], \tag{19.42}$$

when $\sigma_A > k$, $k \in \mathbb{Z}_+$.

Theorem 19.2.5 *Suppose that the process V_k satisfies (19.42) for a set A, and suppose that for deterministic constants $L > M$,*

$$V_k \leq L, \qquad \mathbb{1}\{\sigma_A = k\} V_k \leq M, \qquad k \in \mathbb{Z}_+$$

Then for all $x \in \mathsf{X}$

$$\mathsf{P}_{x_0}\{\sigma_A = \infty\} \geq \frac{V_0(x) - M}{L - M}.$$

Hence if both A and $\{x : V_0(x) > M\}$ lie in $\mathcal{B}^+(\mathsf{X})$ then $\boldsymbol{\Phi}$ is transient.

Proof Define the sequence $\{M_k\}$ by

$$M_{k+1} = V_{k+1}\mathbb{1}\{\sigma_A > k\} + M\mathbb{1}\{\sigma_A \leq k\}.$$

Then, since $\{\sigma_A \leq k\} \in \mathcal{F}_k^{\Phi}$, we have

$$\begin{aligned} \mathsf{E}[M_{k+1} \mid \mathcal{F}_k^{\Phi}] &\geq V_k\mathbb{1}\{\sigma_A > k\} + M\mathbb{1}\{\sigma_A \leq k\} \\ &\geq V_k\mathbb{1}\{\sigma_A > k\} + V_k\mathbb{1}\{\sigma_A = k\} + M\mathbb{1}\{\sigma_A \leq k-1\} \\ &= M_k \end{aligned}$$

and the adapted process $(M_k, \mathcal{F}_k^{\Phi})$ is thus a submartingale. Hence $(L - M_k, \mathcal{F}_k^{\Phi})$ is a positive supermartingale. By Kolmogorov's Inequality (Theorem D.6.3) it follows that for any $T > 0$

$$\mathsf{P}_x\{\sup_{k\geq 0}(L - M_k) \geq T\} \leq \frac{L - M_0(x)}{T}.$$

Letting $T = L - M$, and noting that $M_0(x) \geq V_0(x)$, gives

$$\mathsf{P}_x\{\inf_{k\geq 0} M_k \leq M\} \leq \frac{L - V_0(x)}{L - M}.$$

Finally, since $M_k = M$ for all k sufficiently large whenever $\sigma_A < \infty$, it follows that

$$\mathsf{P}_x\{\sigma_A = \infty\} \geq \mathsf{P}_x\{\inf_{k\geq 0} M_k > M\} \geq \frac{V_0(x) - M}{L - M}$$

which is the desired bound. □

19.2.4 The dependent parameter bilinear model

To illustrate the general results described above we will analyze the dependent parameter bilinear model defined as in (7.23) by the pair of equations

$$\begin{aligned}\theta_{k+1} &= \alpha\theta_k + Z_{k+1}, \qquad |\alpha| < 1\\ Y_{k+1} &= \theta_k Y_k + W_{k+1}\end{aligned}$$

This model is just the simple adaptive control model with the control set to zero; but while the model is somewhat simpler to define than the adaptive control model, we will see that the lack of control makes it much more difficult to show that the model is geometrically ergodic. One of the difficulties with this model is that to date a test function of the form (V4) has not been explicitly computed, though we will show here that a time varying test function of the form (19.39) can be constructed.

The proof will require a substantially more stringent bound on the parameter process than that which was used in the proof of Proposition 17.3.5. We will assume that

$$\zeta_z^2 := \mathsf{E}\Big[\exp\Big\{\frac{2}{1-|\alpha|}|Z_1| - 2\Big\}\Big] < 1. \tag{19.43}$$

Using a history dependent test function of the form (19.39) we will prove the following

Theorem 19.2.6 *Suppose that conditions (DBL1)-(DBL2) hold, and (19.43) is satisfied. Then $\boldsymbol{\Phi}$ is geometrically ergodic, and hence possesses a unique invariant probability π. The CLT and LIL hold for the processes $\mathbf{Y}$ and $\boldsymbol{\theta}$, and for each initial condition $x \in \mathsf{X}$,*

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} Y_k^2 &= \int y^2\, d\pi < \infty \qquad a.s.\ [\mathsf{P}_x]\\ |\mathsf{E}_x[Y_k^2] - \int y^2 d\pi| &\le M(x)\rho^k, \quad k \ge 0\end{aligned}$$

where M is a continuous function on X and $0 < \rho < 1$. □

PROOF It follows as in the proof of Proposition 17.3.5 that the joint process $\Phi_k = \binom{\theta_k}{Y_k}$, $k \ge 0$, is an aperiodic, ψ-irreducible T-chain.

In view of Theorem 19.2.3 it is enough to show that the history dependent drift (19.39) holds for an adapted process $\{V_k\}$. We now indicate how to construct such a process now.

First use the estimate $x \le e^{-1}e^x$ to show

$$|\prod_{i=j}^{k}\theta_i| \le e^{-(k-j+1)}\Big(\prod_{i=j}^{k}\exp|\theta_i|\Big) = e^{-(k-j+1)}\exp\Big(\sum_{i=j}^{k}|\theta_i|\Big). \tag{19.44}$$

But since by (2.13),

$$\sum_{i=j}^{k}|\theta_i| \le |\alpha|\sum_{i=j}^{k}|\theta_i| + |\alpha||\theta_{j-1}| + \sum_{i=j}^{k}|Z_i|,$$

we have

$$\sum_{i=j}^{k} |\theta_i| \leq \frac{|\alpha|}{1-|\alpha|}|\theta_{j-1}| + \frac{1}{1-|\alpha|}\sum_{i=j}^{k}|Z_i|, \tag{19.45}$$

and (19.44) and (19.45) imply the bound, for $j \geq 1$,

$$|\prod_{i=j}^{k}\theta_i| \leq e^{-(k-j+1)}\exp\{\frac{|\alpha|}{1-|\alpha|}|\theta_{j-1}|\} \times \exp\{\frac{1}{1-|\alpha|}\sum_{i=j}^{k}|Z_i|\}. \tag{19.46}$$

Squaring both sides of (17.28) and applying (19.46), we obtain the bound

$$Y_{k+1}^2 \leq 3A_k + 3B_k + 3W_{k+1}^2 \tag{19.47}$$

for all $k \in \mathbb{Z}_+$, where

$$\begin{aligned} A_k &= \{\sum_{j=1}^{k}|W_j|\exp\{\frac{|\alpha|}{1-|\alpha|}|\theta_{j-1}|\}\prod_{i=j}^{k}\exp\{\frac{1}{1-|\alpha|}|Z_i|-1\}\}^2 \\ B_k &= \theta_0^2 Y_0^2 \exp\{\frac{2|\alpha|}{1-|\alpha|}|\theta_0|\}\prod_{i=1}^{k}\exp\{\frac{2}{1-|\alpha|}|Z_i|-2\}. \end{aligned}$$

If we define

$$C_k = \exp\{\frac{2|\alpha|}{1-|\alpha|}|\theta_k|\}$$

we have the three bounds, valid for any $\varepsilon > 0$,

$$\begin{aligned} \mathsf{E}[A_{k+1} \mid \mathcal{F}_k^{\Phi}] &\leq \zeta_z^2\{(1+\varepsilon)A_k + (1+\varepsilon^{-1})\mathsf{E}[W^2]C_k\} \\ \mathsf{E}[B_{k+1} \mid \mathcal{F}_k^{\Phi}] &\leq \zeta_z^2 B_k \\ \mathsf{E}[C_{k+1} \mid \mathcal{F}_k^{\Phi}] &\leq |\alpha|C_k + (1-|\alpha|)(\mathsf{E}[\exp\{\frac{2|\alpha|}{1-|\alpha|}|Z_1|\}])^{\frac{1}{1-|\alpha|}}. \end{aligned}$$

This is shown in [177] and we omit the details which are too lengthy for this exposition. The constant ε will be assumed small, but we will keep it free until we have performed one more calculation. For $k \geq 0$ we make the definition

$$V_k = \varepsilon^3 Y_k^2 + \varepsilon^2 A_k + B_k + C_k.$$

We have for any $k \geq 0$,

$$\varepsilon^3 Y_k^2 + \exp\{\frac{2|\alpha|}{1-|\alpha|}|\theta_k|\} \leq V_k,$$

and since $V(y,\theta) = \varepsilon^3 y^2 + \exp\{\frac{2|\alpha|}{1-|\alpha|}|\theta|\}$ is a norm-like function on X, it follows that the sequence $\{V_k : k \in \mathbb{Z}_+\}$ is norm-like.

Using the bounds above we have for some $R < \infty$,

$$\begin{aligned} \mathsf{E}[V_{k+1} \mid \mathcal{F}_k^{\Phi}] \leq\ & 3\varepsilon^3 A_k^2 + 3\varepsilon^3 B_k + \zeta_z^2\varepsilon^2(1+\varepsilon)A_k + \zeta_z^2\varepsilon^2(1+\varepsilon^{-1})\mathsf{E}[W^2]C_k \\ & + \zeta_z^2 B_k + |\alpha|C_k + R. \end{aligned}$$

Rearranging terms gives

$$\begin{aligned} \mathsf{E}[V_{k+1} \mid \mathcal{F}_k^{\Phi}] \leq\ & \{3\varepsilon + \zeta_z^2(1+\varepsilon)\}\varepsilon^2 A_k + \{3\varepsilon^3 + \zeta_z^2\}B_k \\ & + \{|\alpha| + \varepsilon^2(1+\varepsilon^{-1})\mathsf{E}[W^2]\zeta_z^2\}C_k + R. \end{aligned}$$

Hence (19.39) holds with

$$\lambda = \max(|\alpha| + \zeta_z^2\varepsilon^2(1+\varepsilon^{-1})\mathsf{E}[W^2], \zeta_z^2 + 3\varepsilon^3, \zeta_z^2(1+\varepsilon) + 3\varepsilon),$$

and for ε sufficiently small, we have $\lambda < 1$ as required. □

19.3 Mixed drift conditions

One of the themes of this book has been the interplay between the various stability concepts, and the existence of test functions which give appropriate and consistent drift towards the center of the space.

We conclude with a section which considers chains where the drift is mixed: that is, inward in some parts of the space, and outward in other parts. Of course, it again follows from all we have done to date that for some functions (and in particular the expected hitting time functions V_C) the one step drift will always be towards the set C from initial conditions outside of C. However, it is of considerable intuitive interest to consider the drift when the function V is relatively arbitrary, in which case there is no reason *a priori* to expect that the drift will be consistent in any useful way.

We will find in this section that for a large class of functions, an appropriately averaged drift over the state space is indeed "inwards" when the chain is positive, and "outwards" when the chain is null. This accounts in yet another way for the success of the seemingly simple drift criteria as tools for classifying general chains.

19.3.1 The limiting-average drift

Suppose that V is an everywhere finite non-negative function satisfying

$$\int P(x,dy)|V(y)-V(x)| \le d < \infty, \qquad x \in \mathsf{X}. \tag{19.48}$$

Then we have, for all $n \in \mathbb{Z}_+$, $x \in \mathsf{X}$,

$$\int P^n(x,dy)|\Delta V(y)| \le d$$

and thus the functions

$$n^{-1}\sum_{k=1}^{n}\int_{C^c} P^k(x,dy)\Delta V(y) \tag{19.49}$$

are all well-defined and finite everywhere. Obviously we need a little less than (19.48) to guarantee this, but (19.48) will also be a convenient condition elsewhere.

Theorem 19.3.1 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible, and that $V \ge 0$ satisfies (19.48). A sufficient condition for the chain to be positive is that for some one $x \in \mathsf{X}$ and some petite set C*

$$\liminf_{n\to\infty} n^{-1}\sum_{k=1}^{n}\int_{C^c} P^k(x,dy)\Delta V(y) < 0. \tag{19.50}$$

PROOF By definition we have

$$\begin{aligned}\textstyle\int P^{n+1}(x,dy)V(y) &= \textstyle\int P^n(x,dw)\int P(w,dy)V(y)\\ &= \textstyle\int P^n(x,dy)\Delta V(y) + \int P^n(x,dy)V(y)\end{aligned} \tag{19.51}$$

where all the terms in (19.51) are finite by induction and (19.48). By iteration, we then get

$$n^{-1}\int P^{n+1}(x,dy)V(y)=n^{-1}\sum_{k=1}^{n}\int P^k(x,dy)\Delta V(y)+n^{-1}[\Delta V(x)+V(x)]$$

so that as $n\to\infty$

$$\liminf n^{-1}\sum_{k=1}^{n}\int P^n(x,dy)\Delta V(y)\geq 0. \tag{19.52}$$

Now suppose by way of contradiction that $\boldsymbol{\Phi}$ is null; then from Theorem 18.2.2 we have that the petite set C is null, and so for every x we have by the bound in (19.48)

$$\lim_{n\to\infty}\int_C P^n(x,dy)\Delta V(y)=0.$$

This, together with (19.52), cannot be true when we have assumed (19.50); so the chain is indeed positive. □

There is a converse to this result. We first show that for positive chains and suitable functions V, the drift ΔV, π-averaged over the whole space, is in fact zero.

Theorem 19.3.2 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible, positive with invariant probability measure π, and that $V\geq 0$ satisfies (19.48). Then*

$$\int_{\mathsf{X}}\pi(dy)\Delta V(y)=0. \tag{19.53}$$

PROOF Consider the function $M_z(x)$ defined for $z\in(0,1)$ by

$$M_z(x)=\int P(x,dy)[z^{V(x)}-z^{V(y)}]/[1-z]$$

We first show that $|M_z(x)|$ is uniformly bounded for $x\in\mathsf{X}$ and $z\in(\frac{1}{2},1)$ under the bound (19.48).

By the Mean Value Theorem and non-negativity of V we have for any $0<z<1$,

$$\begin{aligned}|z^{V(x)}-z^{V(y)}| &\leq |V(x)-V(y)|\sup_{t\geq 0}|\frac{d}{dt}z^t| \\ &= |V(x)-V(y)||\log(z)|. \end{aligned}\tag{19.54}$$

Hence under (19.48), for all $x\in\mathsf{X}$ and $z\in(0,1)$,

$$|M_z(x)|\leq\frac{|\log(z)|}{1-z}\int P(x,dy)|V(x)-V(y)|\leq\frac{d}{z} \tag{19.55}$$

which establishes the claimed boundedness of $|M_z(x)|$.

Moreover, by (19.54) and dominated convergence,

$$\lim_{z\uparrow 1}M_z(x)=\int P(x,dy)\Big\{\lim_{z\uparrow 1}\frac{z^{V(x)}-z^{V(y)}}{1-z}\Big\}=\Delta V(x). \tag{19.56}$$

Since $\int\pi(dx)z^{V(x)}<\infty$ for fixed $z\in(0,1)$, we can interchange the order of integration and find

$$\int\pi(dx)M_z(x)=\int\pi(dx)\int P(x,dy)[z^{V(x)}-z^{V(y)}]/[1-z]=0.$$

Hence by the Dominated Convergence Theorem once more we have

$$\begin{aligned} 0 &= \lim_{z\uparrow 1} \int \pi(dx) M_z(x) \\ &= \int \pi(dx)\Big[\lim_{z\uparrow 1} M_z(x)\Big] \qquad (19.57) \\ &= \int \pi(dx) \Delta V(x) \end{aligned}$$

as required. □

Intuitively, one might expect from stationarity that the balance equation (19.53) will hold in complete generality. But we know that this is not the case without some auxiliary conditions such as (19.48): we saw this in Section 11.5.1, where we showed an example of a positive chain with everywhere strictly positive drift.

We now see that the balanced drift of (19.53) occurs, as one might expect from (19.50), from the inward drift towards suitable sets C, combined with an outward drift from such sets. This gives us the converse to Theorem 19.3.1.

Theorem 19.3.3 *Suppose that Φ is ψ-irreducible, and that $V \geq 0$ satisfies (19.48). If C is a sublevel set of V with $C^c, C \in \mathcal{B}^+(\mathsf{X})$, then a necessary condition for the chain to be positive is that*

$$\int_{C^c} \pi(dw) \Delta V(w) < 0 \qquad (19.58)$$

in which case for almost all $x \in \mathsf{X}$

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \int_{C^c} P^k(x, dy) \Delta V(y) < 0. \qquad (19.59)$$

Thus, under these conditions, (19.50) is necessary and sufficient for positivity.

Proof Suppose the chain is positive, and that $C = \{x : V(x) \leq b\} \in \mathcal{B}^+(\mathsf{X})$ is a sublevel set of the function V, so that obviously

$$V(y) > \sup_{x\in C} V(x), \qquad y \in C^c. \qquad (19.60)$$

From (19.48) we certainly have that drift off C is bounded, so that

$$|\Delta V(x)| \leq B' < \infty, \qquad x \in C, \qquad (19.61)$$

and in particular $\int_C \pi(dw) \Delta V(w) \leq B'$.

Using the invariance of π,

$$\begin{aligned} \int_C \pi(dw)\Delta V(w) &= \int_C \pi(dx) \int P(x,dw)V(w) - \int_C \pi(dw)V(w) \\ &= \int_C \pi(dx)[\int_{C^c} P(x,dw)V(w) + \int_C P(x,dw)V(w)] \\ &\quad - \int_C [\int_{\mathsf{X}} \pi(dx)P(x,dw)]V(w) \qquad (19.62) \\ &= \int_C \pi(dx) \int_{C^c} P(x,dw)V(w) \\ &\quad + \int_C \pi(dx) \int_C P(x,dw)V(w) \\ &\quad - \int_{\mathsf{X}} \pi(dx) \int_C P(x,dw)V(w). \end{aligned}$$

Now provided the set C^c is in $\mathcal{B}^+(\mathsf{X})$, we show the right hand side of (19.62) is strictly positive. To see this requires two steps.

First observe that $\int_C \pi(dx)P(x,C^c) > 0$ since $C, C^c \in \mathcal{B}^+(\mathsf{X})$. Since $V(y) > \sup_{w\in C} V(w)$ for $y \in C^c$ we have

$$\int_C \pi(dx) \int_{C^c} P(x,dw)V(w) > \Big(\sup_{w\in C} V(w)\Big) \int_C \pi(dx)P(x,C^c) \tag{19.63}$$

showing from (19.62) that

$$\int_C \pi(dw)\Delta V(w) > \Big(\sup_{w\in C} V(w)\Big)\Big[\int_C \pi(dx)P(x,C^c) - \int_{C^c} \pi(dx)P(x,C)\Big]. \tag{19.64}$$

Secondly, we have the balanced-flow equation

$$\begin{aligned} \textstyle\int_C \pi(dx)P(x,C^c) &= \textstyle\int_C \pi(dx)[1-P(x,C)] \\ &= \textstyle\pi(C) - \int_C \pi(dx)P(x,C) \\ &= \textstyle\int_{\mathsf{X}} \pi(dx)P(x,C) - \int_C \pi(dx)P(x,C) \\ &= \textstyle\int_{C^c} \pi(dx)P(x,C). \end{aligned} \tag{19.65}$$

Putting this into the strict inequality in (19.64), we have that

$$\int_C \pi(dw)\Delta V(w) > 0 \tag{19.66}$$

provided that V does not vanish on C. If V does vanish on C then (19.66) holds automatically.

But now, under (19.48) we have $\int \pi(dx)\Delta V(x) = 0$ from (19.53), and so (19.58) is a consequence of this and (19.66). Since $\Delta V(y)$ is bounded under (19.48), (19.59) is actually identical to (19.58) and the theorem is proved. □

These results show that for a wide class of functions, our criteria for positivity and nullity, given respectively in Section 11.3 and Section 11.5.1, are essentially the two extreme cases of this mixed-drift result. We conclude with an example where similar mixed behavior may be exhibited quite explicitly.

19.3.2 A mixed drift criterion for stability of the ladder chain

We return to the ladder chain defined by (10.38). Recall that the structure of the stationary measure, when it exists, is known to have an operator-geometric form as in Section 10.5.3. Here we consider conditions under which such a stationary measure exists.

If we assume that the zero-level transitions have the form

$$\Lambda_i^*(x,A) = P(i,x;0,A) = \sum_{j=k+1}^{\infty} \Lambda_j(x,A) \tag{19.67}$$

so that there is a greater degree of homogeneity than in the general model, then the operator

$$\Lambda(x,A) := \sum_{j=0}^{\infty} \Lambda_j(x,A)$$

is stochastic.

Thus $\Lambda(x,A)$ defines a Markov chain $\boldsymbol{\Phi}^{\Lambda}$, which is the marginal position of $\boldsymbol{\Phi}$ ignoring the actual rung: by direct calculation we can check that for any B

$$P^n(i,x;\mathbb{Z}_+ \times B) = \Lambda^n(x,B). \tag{19.68}$$

Moreover, (19.68) immediately gives that if $\boldsymbol{\Phi}$ is ψ-irreducible, then $\boldsymbol{\Phi}^{\Lambda}$ is ψ^*-irreducible, where $\psi^*(B) = \psi(\mathbb{Z}_+ \times B)$.

Now define, for any $w \in \mathsf{X}$, the expected change in ladder height by

$$\beta(w) = \sum_{j=0}^{\infty} j\Lambda_j(x,\mathsf{X}) : \tag{19.69}$$

if $\beta(w) > 1+\delta$ for all w then, exactly as in our analysis of the random walk on a half line, we have that

$$\mathsf{E}_{(i,w)}[\tau_C] < \infty$$

for all $i > M, w \in \mathsf{X}$, where $C = \cup_0^M \{j \times \mathsf{X}\}$ is the "bottom end" of the ladder.

But one might not have such downwards drift uniform across the rungs. The result we prove is thus an average drift criterion.

Theorem 19.3.4 *Suppose that the chain $\boldsymbol{\Phi}$ is ψ-irreducible, and has the structure (19.67). If the marginal chain $\boldsymbol{\Phi}^{\Lambda}$ admits an invariant probability measure ν such that*

$$\int \nu(dw)\beta(w) > 1 \tag{19.70}$$

then $\boldsymbol{\Phi}$ admits an invariant probability measure π.

Proof The proof is similar to that of Theorem 19.3.1, but we do not assume boundedness of the drifts so we must be a little more delicate. Choosing $V(i,w) = i$, we have first that

$$\Delta V(i,w) = 1 - \sum_{j=0}^{i} j\Lambda_j(x,\mathsf{X}) - (i+1)\sum_{j=i+1}^{\infty} \Lambda_j(x,\mathsf{X});$$

note that in particular for $i > d$ this gives

$$\Delta V(i,w) \le \Delta V(d,w), \qquad w \in \mathsf{X}. \tag{19.71}$$

Now even though (19.48) is not assumed, because $|\Delta V(i,w)| \le d+1$ for $i \le d$, and because starting at level i, after k steps the chain cannot be above level $i+k$, we see exactly as in proving (19.52) that

$$\liminf n^{-1} \sum_{k=1}^{n} \int \sum_j P^k(i,x;j \times dy)\Delta V(j,y) \ge 0. \tag{19.72}$$

We now show that this average non-negative drift is not possible under (19.70), unless the chain is positive.

From (19.70) we have

$$0 > \lim_{k\to\infty} \int \nu(dw)\Delta V(k,w). \tag{19.73}$$

Choose d sufficiently large that

$$0 > \int \nu(dw)\Delta V(d,w). \tag{19.74}$$

Further truncate by choosing $v \geq 1$ large enough that if $D_v = \{y : \Delta V(d,y) \geq -v\}$ then, using (19.74)

$$0 > \int_{D_v} \nu(dw)\Delta V(d,w). \tag{19.75}$$

Now decompose the left hand side of (19.72) as

$$\begin{aligned}
n^{-1}\sum_{k=1}^{n}\int_{\mathsf{X}}\sum_{j} P^k(i,x;j\times dy)\Delta V(j,y) \\
&= n^{-1}\sum_{k=1}^{n}\int_{\mathsf{X}}\sum_{j=0}^{d-1} P^k(i,x;j\times dy)\Delta V(j,y) \\
&\quad + n^{-1}\sum_{k=1}^{n}\int_{\mathsf{X}}\sum_{j\geq d} P^k(i,x;j\times dy)\Delta V(j,y) \\
&\leq n^{-1}\sum_{k=1}^{n} d\sum_{j=0}^{d-1} P^k(i,x;j\times \mathsf{X}) \\
&\quad + n^{-1}\sum_{k=1}^{n}\int_{D_v}\sum_{j\geq d} P^k(i,x;j\times dy)\Delta V(j,y)
\end{aligned} \tag{19.76}$$

since on D_v^c we have $\Delta V(d,y) \leq -1$.

Assume the chain is not positive: we now show that (19.76) is strictly negative, and this provides the required contradiction of (19.72).

We know from Theorem 18.2.2 that there exists a sequence C_n of null sets with $C_n \uparrow \mathbb{Z}_+ \times \mathsf{X}$.

In fact, in this model we now show that every rung is such a null set. Fix a rung $j\times\mathsf{X}$, and let $C_n(j) = C_n \cap j\times\mathsf{X}$. Since Φ is assumed ψ^*-irreducible with an invariant probability measure ν, we have from the ergodic theorem (13.63) that for ψ^*-a.e x, and any M,

$$\lim n^{-1}\sum_{k=1}^{n} \Lambda^k(x, C_M(j)) = \nu(C_M(j)).$$

Choose M so large that $\nu(C_M(j)) \geq 1-\varepsilon$ for a given $\varepsilon > 0$. Then we have

$$\begin{aligned}
\lim n^{-1}\textstyle\sum_{k=1}^{n} P^k(i,x;j\times\mathsf{X}) &= \lim n^{-1}\textstyle\sum_{k=1}^{n} P^k(i,x;j\times C_M(j)) \\
&\quad + \lim n^{-1}\textstyle\sum_{k=1}^{n} P^k(i,x;j\times [C_M(j)]^c) \\
&\leq \lim n^{-1}\textstyle\sum_{k=1}^{n} P^k(i,x;C_M) \\
&\quad + \lim n^{-1}\textstyle\sum_{k=1}^{n} \Lambda^k(x,[C_M(j)]^c) \\
&\leq \varepsilon
\end{aligned} \tag{19.77}$$

which shows the rung $j \times \mathsf{X}$ to be null as claimed.

Using (19.77) we have in particular that for any B, and d as above,

$$\begin{aligned} \nu(B) &= \lim n^{-1} \textstyle\sum_{k=1}^{n} \Lambda^k(x,B) \\ &= \lim n^{-1} \textstyle\sum_{k=1}^{n} \sum_{j=0}^{d-1} P^k(i,x;j\times B) \\ &\qquad + \lim n^{-1} \textstyle\sum_{k=1}^{n} \sum_{j=d}^{\infty} P^k(i,x;j\times B) \\ &= \lim n^{-1} \textstyle\sum_{k=1}^{n} \sum_{j=d}^{\infty} P^k(i,x;j\times B). \end{aligned} \tag{19.78}$$

We now use (19.77) and (19.78) in (19.76). This gives, successively,

$$\begin{aligned} &\liminf_{n\to\infty} n^{-1} \sum_{k=1}^{n} \int_{\mathsf{X}} \sum_{j} P^k(i,x;j\times dy)\Delta V(j,y) \\ &\le \liminf_{n\to\infty} n^{-1} \textstyle\sum_{k=1}^{n} \int_{D_v} \sum_{j\ge d} P^k(i,x;j\times dy)\Delta V(j,y) \\ &= \textstyle\int_{D_v} \nu(dy)\Delta V(j,y) < 0 \end{aligned} \tag{19.79}$$

from the construction in (19.75).

This is the required contradiction of (19.72) and we are finished. □

It is obviously of interest to know whether the same average drift condition suffices for positivity when (19.67) does not hold.

In general, this is a subtle question. Writing as before $[0] = 0 \times \mathsf{X}$, we obviously have that under (19.70)

$$\mathsf{E}_{0,y}[\tau_{[0]}] < \infty \tag{19.80}$$

for ν-a.e. y, since this quantity does not depend on the detailed hitting distribution on $[0]$. But although this ensures that the process on $[0]$ is well-defined, it does not even ensure that it is recurrent.

As an example of the range of behaviors possible, let us take $\mathsf{X} = \mathbb{Z}_+$ also, and consider a chain that can move only up one rung or down one rung: specifically, choose $0 < p, q < 1$ and

$$\begin{array}{rcllcl} \Lambda_0(x,x-1) &=& pq, & x &\ge& 1 \\ \Lambda_0(x,x+1) &=& (1-p)q, & x &\ge& 0 \\ \Lambda_2(x,x-1) &=& p(1-q), & x &\ge& 1 \\ \Lambda_2(x,x+1) &=& (1-p)(1-q), & x &\ge& 0 \end{array} \tag{19.81}$$

with the transitions on the boundary given by

$$\begin{array}{rcl} \Lambda_0(0,0) &=& pq, \\ \Lambda_2(0,0) &=& p(1-q). \end{array} \tag{19.82}$$

The marginal chain $\boldsymbol{\Phi}^{\Lambda}$ is a random walk on the half line $\{0, 1, \dots\}$ with an invariant measure ν if and only if $p > 1/2$. On the other hand, $\beta(x) > 1$ if and only if $q < 1/2$. Thus (19.70) holds if $q < 1/2 < p$.

This chain falls into the class that we have considered in Theorem 19.3.4; but other behaviors follow if we vary the structure at the bottom rung.

Let us then specify the boundary conditions in a manner other than (19.67): put $\Lambda_1^*(x, x-1) = p(1-q)$ and $\Lambda_1^*(x, x+1) = (1-p)(1-q)$ but

$$\begin{array}{rclcrcl} \Lambda_0^*(x, x-1) & = & r(1-q), & & x & \geq & 1 \\ \Lambda_0^*(x, x+1) & = & (1-r)(1-q), & & x & \geq & 1. \end{array} \tag{19.83}$$

where $0 < r < 1$.

Consider now the expected increments in the chain $\boldsymbol{\Phi}^{[0]}$ on $[0]$. By considering whether the chain leaves $[0]$ or not we have for all $x \geq 1$

$$\mathsf{E}[\Phi_n^{[0]} \mid \Phi_{n-1}^{[0]} = x] - x \geq (1-2r)(1-q) + (1-2p)\Big(\frac{1-q}{1-2q} + 1\Big)q : \tag{19.84}$$

here the second term follows since, on an excursion from $[0]$, the expected drift to the left at every step is no more than $(1-2p)$ independent of level change, and the expected number of steps to return to $[0]$ from $1 \times \mathsf{X}$ is $(1-q)/(1-2q)$.

From (19.84) we therefore have that the chain $\boldsymbol{\Phi}^{[0]}$ is transient if r and q are small enough, and $p - 1/2$ is not too large.

This example shows the critical need to identify petite sets and the return times to them in classifying any chain: here we have an example where the set $[0]$ is not petite, although it has many of the properties of a petite set. Yet even though we have (19.80) proven, we do not even have enough to guarantee the chain is recurrent.

19.3.3 Stability of the GI/G/1 queue

We saw in Section 3.5 that with appropriate choice of kernels the ladder chain serves as a model for the GI/G/1 queue. We will use the average drift condition of Theorem 19.3.4 to derive criteria for stability of this model.

Of course, in this case we do not have (19.67), and the example at the end of the last section shows that we cannot necessarily deduce anything from (19.70).

In this case, however, we have as in Section 10.5.3 that $[0]$ is petite, and that the process on $[0]$, if honest, has invariant measure H where H is the service time distribution. If we can satisfy (19.70), then, it follows from (19.80) that the process on $[0]$ is indeed honest, and we only have to check further that

$$\int H(dy)\mathsf{E}_{0,y}[\tau_{[0]}] < \infty \tag{19.85}$$

to derive positivity.

We conclude by proving through this approach a result complementing the result found in quite another way in Proposition 11.4.4.

Theorem 19.3.5 *The GI/G/1 queue with mean inter-arrival time λ and mean service time μ satisfies (19.70) if and only if $\lambda > \mu$, and in this case the chain has an invariant measure given by (10.53).*

PROOF From the representations (3.43) and (3.44), we have that the kernel

$$\Lambda(x, [0, y]) = \int_0^\infty G(dt)P^t(x, y)$$

where $P^t(x,y) = \mathsf{P}(R_t \le y \mid R_0 \le y)$ is the forward recurrence time process in a renewal process $N(t)$ generated by increments with distribution H.

Since H has finite mean μ, we know from (10.37) that $P^\delta(x,y)$ has invariant measure

$$\nu[0,x] = \mu^{-1}\int_0^x [1-H(x)]dx$$

for every δ: thus ν is also invariant for Λ.

On the other hand, from (3.43),

$$\begin{aligned}\beta(x) &= \sum_{n=0}^\infty n\Lambda_n(x,[0,\infty)) \\ &= \sum_{n=0}^\infty n\int G(dt)P_n^t(x,\infty) \\ &= \int G(dt)\mathsf{E}[N(t) \mid R_0 = x].\end{aligned}$$

The stationarity of ν for the renewal process $N(t)$ shows that

$$\int_0^\infty \nu(dx)\mathsf{E}[N(t) \mid R_0 = x] = t/\mu$$

and so by Fubini's Theorem, we therefore have

$$\begin{aligned}\textstyle\int \nu(dx)\beta(x) &= \textstyle\int_0^\infty \Big[\int_0^\infty \nu(dx)\mathsf{E}[N(t) \mid R_0 = x]\Big] G(dt) \\ &= \textstyle\int_0^\infty [t/\mu]G(dt) \\ &= \lambda/\mu\end{aligned} \tag{19.86}$$

which proves the first part of the theorem.

To conclude, we note that in this particular case, we know more about the structure of $\mathsf{E}_{0,y}[\tau_{[0]}]$, and this enables us to move from the case where (19.67) holds. Given the starting configuration $(0,y)$, let n_y denote the number of customers arriving in the first service time y: if $\eta(\le\infty)$ denotes the expected number of customers in a busy period of the queue, then by using the trick of rearranging the order of service to deal with each of the identical n_y "busy periods" generated by these customers separately, we have the linear structure

$$\mathsf{E}_{0,y}[\tau_{[0]}] = 1 + \mathsf{E}_{0,y}[n_y\eta] = 1 + \eta\sum_{n=0}^\infty G^{n*}[0,y]. \tag{19.87}$$

As in (19.80), we at least know that since (19.70) holds, the left hand side of this equation is finite, so that $\eta < \infty$. Moreover, from the Blackwell Renewal Theorem (Theorem 14.5.1) we have for any ε and large y

$$\sum_{n=0}^\infty G^{n*}[0,y] \le y[\lambda^{-1}+\varepsilon] \tag{19.88}$$

so that, finally, (19.85) follows from (19.87), (19.88), and the fact that the mean of H is finite. □

19.4 Commentary

Despite the success of the simple drift, or Foster-Lyapunov, approach there is a growing need for more subtle variations such as those we present here.

There are several cases in the literature where the analysis of state-dependent (or at least not simple one-step) drift appears unavoidable: see Tjøstheim [265] or Chen and Tsay [46], where m-step skeletons $\{\Phi_{mk}\}$ are analyzed. Analysis of this kind is simplified if the various parts of the space can be considered separately as in Section 19.1.2.

In the countable space context, Theorem 19.1.1 was first shown as Theorem 1.3 and Theorem 19.1.2 as Theorem 1.4 of Malyšhev and Men'šikov [159]. Their proofs, especially of Theorem 19.1.2, are more complex than those based on sample path arguments, which were developed along with Theorem 19.1.3 in [184]. As noted there, the result can be extended by choosing $n(x)$ as a random variable, conditionally independent of the process, on $\mathbb{Z}_+$. In the special case where $n(x)$ has a uniform distribution on $[1, n]$ independent of x, we get a time averaged result used by Meyn and Down [175] in analyzing stability of queueing networks. If the variable has a point mass at $n(x)$ we get the results given here.

Models of random walk on the orthant in Section 19.1.2 have been analyzed in numerous different ways on the integer quadrant $\mathbb{Z}_+^2$ by, for example, [160, 167, 159, 230, 72]. Much of their work pertains to more general models which assume different drifts on the boundary, thus leading to more complex conditions. In [160, 167, 159] it is assumed that the increments are bounded (although they also analyze higher dimensional models), whilst in [230, 72] it is shown that one can actually choose $n = 1$ if a quadratic function is used for a test function, whilst weakening the bounded increments assumption to a second moment condition: this method appears to go back to Kingman [135].

As we have noted, positive recurrence in the simple case illustrated here could be established more easily given the independence of the two components. However, the bound using linear functions in (19.25) seems to be new, as does the continuous space methodology we use here.

The antibody model here is based on that in [184]. The attack pattern of the "invaders" is modeled to a large extent on the rabies model developed in Bartoszyński [16], although the need to be the same order of magnitude as the antibody group is a weaker assumption than that implicit in the continuous time continuous space model there.

The results in Section 19.2 are largely taken from Meyn and Tweedie [178]: they appear to give a fruitful approach to more complex models, and the seeming simplicity of the presentation here is largely a function of the development of the methods based on Dynkin's formula for the non-time varying case. An application to adaptive control is given in Meyn and Guo [176], where drift functions which depend on the whole history of the chain are used systematically. Regrettably, examples using this approach are typically too complex to present here.

The dependent parameter bilinear time series model is analyzed in [177], from which we adopt the proof of Theorem 19.2.6. In Karlsen [123] a decoupling inequality of [137] is used to obtain a second order stationary solution in the Gaussian parameter case, and Brandt [28] provides a simple argument, similar to the proof of

Proposition 17.3.4, to obtain boundedness in probability for general bilinear time series models with stationary coefficients.

Results on mixed drifts, such as those in Section 19.3.1, have been discovered independently several times.

Although Neuts [193] analyzed a two-drift chain in detail, on a countable space the first approach to classifying chains with different drifts appears to be due to Marlin [163]. He considered the special case of $V(x) = x$ and assumed a fixed finite number of different drifts. The form given here was developed for countable spaces by Tweedie [274] (although the proof there is incomplete) and Rosberg [226], who gives a slightly different converse statement. A general state space form is in Tweedie [276].

The condition (19.55) for the converse result to hold, and which also suffices to ensure that $\Delta V(w) \geq 0$ on C^c implies non-positivity, is known as Kaplan's condition [121]: the general state space version sketched here is adapted from a countable space version in [235]. Related results are in [261].

The average mean drift criterion for the ladder process in Section 19.3.2 is due to Neuts [194] when the rungs are finite, and is proved there by matrix methods: the general result is in [277], and (19.70) is also shown there to be necessary for positivity under reasonable assumptions.

The final criterion for stability of the GI/G/1 queue produced by this analysis is of course totally standard [10]: that the very indirect Markovian approach reproduces this result exactly brings us to a remarkably reassuring conclusion.

Added in Second Printing In the past year, Dai has shown in [57] that the state-dependent drift criterion Theorem 19.1.2 leads to a new approach to the stability of stochastic queueing network models via the analysis of a simpler deterministic fluid model. Related work has been developed by Chen [45] and Stolyar [257], and these results have been strengthened in Dai and Weiss [59] and Dai and Meyn [58].