15

Geometric Ergodicity

The previous two chapters have shown that for positive Harris chains, convergence of $\mathsf{E}_x[f(\Phi_k)]$ is guaranteed from almost all initial states x provided only $\pi(f) < \infty$. Strong though this is, for many models used in practice even more can be said: there is often a rate of convergence ρ such that

$$\|P^n(x, \cdot\,) - \pi\|_f = o(\rho^n)$$

where the rate $\rho < 1$ can be chosen essentially independent of the initial point x.

The purpose of this chapter is to give conditions under which convergence takes place at such a uniform geometric rate. Because of the power of the final form of these results, and the wide range of processes for which they hold (which include many of those already analyzed as ergodic) it is not too strong a statement that this "geometrically ergodic" context constitutes the most useful of all of those we present, and for this reason we have devoted two chapters to this topic.

The following result summarizes the highlights of this chapter, where we focus on bounds such as (15.4) and the strong relationship between such bounds and the drift criterion given in (15.3). In Chapter 16 we will explore a number of examples in detail, and describe techniques for moving from ergodicity to geometric ergodicity. The development there is based primarily on the results of this chapter, and also on an interpretation of the geometric convergence (15.4) in terms of convergence of the kernels $\{P^k\}$ in a certain induced operator norm.

Theorem 15.0.1 (Geometric Ergodic Theorem) *Suppose that the chain Φ is ψ-irreducible and aperiodic. Then the following three conditions are equivalent:*

(i) *The chain Φ is positive recurrent with invariant probability measure π, and there exists some ν-petite set $C \in \mathcal{B}^+(\mathsf{X})$, $\rho_C < 1$ and $M_C < \infty$, and $P^\infty(C) > 0$ such that for all $x \in C$*

$$|P^n(x, C) - P^\infty(C)| \le M_C \rho_C^n. \tag{15.1}$$

(ii) *There exists some petite set $C \in \mathcal{B}(\mathsf{X})$ and $\kappa > 1$ such that*

$$\sup_{x \in C} \mathsf{E}_x[\kappa^{\tau_C}] < \infty. \tag{15.2}$$

(iii) *There exists a petite set C, constants $b < \infty$, $\beta > 0$ and a function $V \ge 1$ finite at some one $x_0 \in \mathsf{X}$ satisfying*

$$\Delta V(x) \le -\beta V(x) + b\mathbb{1}_C(x), \qquad x \in \mathsf{X}. \tag{15.3}$$

Any of these three conditions imply that the set $S_V = \{x : V(x) < \infty\}$ is absorbing and full, where V is any solution to (15.3) satisfying the conditions of (iii), and there then exist constants $r > 1$, $R < \infty$ such that for any $x \in S_V$

$$\sum_n r^n \|P^n(x,\,\cdot\,) - \pi\|_V \le RV(x). \tag{15.4}$$

PROOF The equivalence of the local geometric rate of convergence property in (i) and the self-geometric recurrence property in (ii) will be shown in Theorem 15.4.3.

The equivalence of the self-geometric recurrence property and the existence of solutions to the drift equation (15.3) is completed in Theorems 15.2.6 and 15.2.4. It is in Theorem 15.4.1 that this is shown to imply the geometric nature of the V-norm convergence in (15.4), while the upper bound on the right hand side of (15.4) follows from Theorem 15.3.3. □

The notable points of this result are that we can use the same function V in (15.4), which leads to the operator norm results in the next chapter; and that the rate r in (15.4) can be chosen independently of the initial starting point.

We initially discuss conditions under which there exists for some $x \in \mathsf{X}$ a rate $r > 1$ such that

$$\|P^n(x,\,\cdot\,) - \pi\|_f \le M_x r^{-n} \tag{15.5}$$

where $M_x < \infty$. Notice that we have introduced f-norm convergence immediately: it will turn out that the methods are not much simplified by first considering the case of bounded f. We also have another advantage in considering geometric rates of convergence compared with the development of our previous ergodicity results. We can exploit the useful fact that (15.5) is equivalent to the requirement that for some $\bar{r}$, $\bar{M}_x$,

$$\sum_n \bar{r}^n \|P^n(x,\,\cdot\,) - \pi\|_f \le \bar{M}_x. \tag{15.6}$$

Hence it is without loss of generality that we will immediately move also to consider the summed form as in (15.6) rather than the n-step convergence as in (15.5).

f-Geometric Ergodicity

We shall call $\boldsymbol{\Phi}$ *f-geometrically ergodic*, where $f \ge 1$, if $\boldsymbol{\Phi}$ is positive Harris with $\pi(f) < \infty$ and there exists a constant $r_f > 1$ such that

$$\sum_{n=1}^{\infty} r_f^n \|P^n(x,\,\cdot\,) - \pi\|_f < \infty \tag{15.7}$$

for all $x \in \mathsf{X}$. If (15.7) holds for $f \equiv 1$ then we call $\boldsymbol{\Phi}$ *geometrically ergodic*.

The development in this chapter follows a pattern similar to that of the previous two chapters: first we consider chains which possess an atom, then move to aperiodic chains via the Nummelin splitting.

This pattern is now well-established: but in considering geometric ergodicity, the extra complexity in introducing both unbounded functions f and exponential moments of hitting times leads to a number of different and sometimes subtle problems. These make the proofs a little harder in the case without an atom than was the situation with either ergodicity or f-ergodicity. However, the final conclusion in (15.4) is well worth this effort.

15.1 Geometric properties: chains with atoms

15.1.1 Using the Regenerative Decomposition

Suppose in this section that $\boldsymbol{\Phi}$ is a positive Harris recurrent chain and that we have an accessible atom $\boldsymbol{\alpha}$ in $\mathcal{B}^+(\mathsf{X})$: as in the previous chapter, we do not consider completely countable spaces separately, as one atom is all that is needed. We will again use the Regenerative Decomposition (13.48) to identify the bounds which will ensure that the chain is f-geometrically ergodic.

Multiplying (13.48) by r^n and summing, we have that

$$\sum_n \|P^n(x,\,\cdot\,) - \pi\|_f\, r^n$$

is bounded by the three sums

$$\begin{array}{c} \displaystyle\sum_{n=1}^{\infty} \int {}_{\alpha}P^n(x,dw) f(w)\, r^n \\[2ex] \displaystyle\pi(\boldsymbol{\alpha}) \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} t_f(j)\, r^n \\[2ex] \displaystyle\sum_{n=1}^{\infty} |a_x * u - \pi(\boldsymbol{\alpha})| * t_f\,(n)\, r^n \end{array} \tag{15.8}$$

Now using Lemma D.7.2 and recalling that $t_f(n) = \int {}_{\alpha}P^n(\boldsymbol{\alpha},dw)f(w)$, we have that the three sums in (15.8) can be bounded individually through

$$\sum_{n=1}^{\infty} \int {}_{\alpha}P^n(x,dw) f(w) r^n \;\leq\; \mathsf{E}_x\Big[\sum_{n=1}^{\tau_\alpha} f(\Phi_n) r^n\Big], \tag{15.9}$$

$$\pi(\boldsymbol{\alpha}) \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} t_f(j) r^n \;\leq\; \frac{r}{r-1}\mathsf{E}_\alpha\Big[\sum_{n=1}^{\tau_\alpha} f(\Phi_n) r^n\Big], \tag{15.10}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} |a_x * u - \pi(\boldsymbol{\alpha})| * t_f\,(n) r^n \\ &\quad= \Big(\textstyle\sum_{n=1}^{\infty} |a_x * u\,(n) - \pi(\boldsymbol{\alpha})| r^n\Big)\Big(\textstyle\sum_{n=1}^{\infty} t_f(n) r^n\Big) \qquad (15.11) \\ &\quad= \Big(\textstyle\sum_{n=1}^{\infty} |a_x * u\,(n) - \pi(\boldsymbol{\alpha})| r^n\Big)\Big(\mathsf{E}_\alpha\Big[\textstyle\sum_{n=1}^{\tau_\alpha} f(\Phi_n) r^n\Big]\Big). \end{aligned}$$

In order to bound the first two sums (15.9) and (15.10), and the second term in the third sum (15.11), we will require an extension of the notion of regularity, or more exactly of f-regularity. For fixed $r \geq 1$ recall the generating function defined in (8.23) for $r < 1$ by

$$U_{\alpha}^{(r)}(x, f) := \mathsf{E}_x\Big[\sum_{n=1}^{\tau_\alpha} f(\Phi_n) r^n\Big]; \tag{15.12}$$

clearly this is defined but possibly infinite for $r \geq 1$. From the inequalities (15.9)-(15.11) above it is apparent that when $\boldsymbol{\Phi}$ admits an accessible atom, establishing f-geometric ergodicity will require finding conditions such that $U_{\alpha}^{(r)}(x, f)$ is finite for some $r > 1$.

The first term in the right hand side of (15.11) can be reduced further. Using the fact that

$$\begin{aligned} |a_x * u\,(n) - \pi(\boldsymbol{\alpha})| &= |a_x * (u - \pi(\boldsymbol{\alpha}))\,(n) - \pi(\boldsymbol{\alpha}) \sum_{j=n+1}^{\infty} a_x(j)| \\ &\leq a_x * |(u - \pi(\boldsymbol{\alpha}))|\,(n) + \pi(\boldsymbol{\alpha}) \sum_{j=n+1}^{\infty} a_x(j) \end{aligned}$$

and again applying Lemma D.7.2, we find the bound

$$\begin{aligned} \sum_{n=1}^{\infty} |a_x * u - \pi(\boldsymbol{\alpha})| r^n &\leq \Big(\sum_{n=1}^{\infty} a_x(n) r^n\Big)\Big(\sum_{n=1}^{\infty} |u(n) - \pi(\boldsymbol{\alpha})| r^n\Big) \\ &\quad + \pi(\boldsymbol{\alpha}) \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} a_x(j) r^n \\ &\leq \Big(\mathsf{E}_x[r^{\tau_\alpha}]\Big)\Big(\sum_{n=1}^{\infty} |u(n) - \pi(\boldsymbol{\alpha})| r^n\Big) + \frac{r}{r-1} \mathsf{E}_x[r^{\tau_\alpha}]. \end{aligned}$$

Thus from (15.9)-(15.11) we might hope to find that convergence of P^n to π takes place at a geometric rate provided

(i) the atom itself is geometrically ergodic, in the sense that

$$\sum_{n=1}^{\infty} |u(n) - \pi(\boldsymbol{\alpha})| r^n$$

converges for some $r > 1$;

(ii) the distribution of τ_α possess an "f-modulated" geometrically decaying tail from both $\boldsymbol{\alpha}$ and from the initial state x, in the sense that both $U_{\alpha}^{(r)}(\boldsymbol{\alpha}, f) < \infty$ and $U_{\alpha}^{(r)}(x, f) < \infty$ for some $r = r_x > 1$: and if we can choose such an r independent of x then we will be able to assert that the overall rate of convergence in (15.4) is also independent of x.

We now show that as with ergodicity or f-ergodicity, a remarkable degree of solidarity in this analysis is indeed possible.

15.1.2 Kendall's Renewal Theorem

As in the ergodic case, we need a key result from renewal theory. Kendall's Theorem shows that for atoms, geometric ergodicity and geometric decay of the tails of the return time distribution are actually equivalent conditions.

Theorem 15.1.1 (Kendall's Theorem) *Let $u(n)$ be an ergodic renewal sequence with increment distribution $p(n)$, and write $u(\infty) = \lim_{n\to\infty} u(n)$. Then the following three conditions are equivalent:*

(i) *There exists $r_0 > 1$ such that the series*

$$U_0(z) := \sum_{n=0}^{\infty} |u(n) - u(\infty)| z^n \tag{15.13}$$

converges for $|z| < r_0$.

(ii) *There exists $r_0 > 1$ such that the function $U(z)$ defined on the complex plane for $|z| < 1$ by*

$$U(z) := \sum_{n=0}^{\infty} u(n) z^n$$

has an analytic extension in the disc $\{|z| < r_0\}$ except for a simple pole at $z = 1$.

(iii) *there exists $\kappa > 1$ such that the series $P(z)$*

$$P(z) := \sum_{n=0}^{\infty} p(n) z^n \tag{15.14}$$

converges for $\{|z| < \kappa\}$.

PROOF Assume that (i) holds. Then by construction the function $F(z)$ defined on the complex plane by

$$F(z) := \sum_{n=0}^{\infty} (u(n) - u(n-1)) z^n$$

has no singularities in the disc $\{|z| < r_0\}$, and since

$$F(z) = (1-z)U(z), \qquad |z| < 1, \tag{15.15}$$

we have that $U(z)$ has no singularities in the disc $\{|z| < r_0\}$ except a simple pole at $z = 1$, so that (ii) holds.

Conversely suppose that (ii) holds. We can then also extend $F(z)$ analytically in the disc $\{|z| < r_0\}$ using (15.15). As the Taylor series expansion is unique, necessarily $F(z) = \sum_{n=0}^{\infty} (u(n) - u(n-1)) z^n$ throughout this larger disc, and so by virtue of Cauchy's inequality

$$\sum_n |u(n) - u(n-1)| r^n < \infty, \qquad r < r_0.$$

Hence from Lemma D.7.2

$$\begin{aligned}\infty &> \sum_n \sum_{m\geq n} |u(m+1)-u(m)|r^n \\ &\geq \sum_n |\sum_{m\geq n} (u(m+1)-u(m))|r^n \\ &= \sum_n |u(\infty)-u(n)|r^n\end{aligned}$$

so that (i) holds.

Now suppose that (iii) holds. Since $P(z)$ is analytic in the disc $\{|z| < \kappa\}$, for any $\varepsilon > 0$ there are at most finitely many values of z such that $P(z) = 1$ in the smaller disc $\{|z| < \kappa - \varepsilon\}$.

By aperiodicity of the sequence $\{p(n)\}$, we have $p(n) > 0$ for all $n > N$ for some N, from Lemma D.7.4. This implies that for $z \neq 1$ on the unit circle $\{|z| = 1\}$, we have

$$\sum_N^\infty p(n)\Re(z^n) < \sum_N^\infty p(n),$$

so that

$$\Re P(z) \leq \sum_0^\infty p(n)\Re(z^n) < \sum_0^\infty p(n) = 1.$$

Consequently only one of these roots, namely $z = 1$, lies on the unit circle, and hence there is some r_0 with $1 < r_0 \leq \kappa$ such that $z = 1$ is the only root of $P(z) = 1$ in the disc $\{|z| < r_0\}$.

Moreover this is a simple root at $z = 1$, since

$$\lim_{z\to 1} \frac{1-P(z)}{1-z} = \frac{d}{dz}P(z)|_{z=1} = \sum np(n) \neq 0.$$

Now the renewal equation (8.12) shows that

$$U(z) = [1 - P(z)]^{-1}$$

is valid at least in the disc $\{|z| < 1\}$, and hence

$$F(z) = (1-z)U(z) = (1-z)[1-P(z)]^{-1} \tag{15.16}$$

has no singularities in the disc $\{|z| < r_0\}$; and so (ii) holds.

Finally, to show that (ii) implies (iii) we again use (15.16): writing this as

$$P(z) = [F(z) - 1 + z]/F(z)$$

shows that $P(z)$ is a ratio of analytic functions and so is itself analytic in the disc $\{|z| < \kappa\}$, where now κ is the first zero of $F(z)$ in $\{|z| < r_0\}$; there are only finitely many such zeros and none of them occurs in the closed unit disc $\{|z| \leq 1\}$ since $P(z)$ is bounded in this disc, so that $\kappa > 1$ as required. □

It would seem that one should be able to prove this result, not only by analysis but also by a coupling argument as in Section 13.2. Clearly one direction of this is easy: if the renewal times are geometric then one can use coupling to get geometric convergence. The other direction does seem to require analytic tools to the best of our knowledge, and so we have given the classical proof here.

15.1.3 The Geometric Ergodic Theorem

Following this result we formalize some of the conditions that will obviously be required in developing a geometric ergodicity result.

Kendall Atoms and Geometrically Ergodic Atoms

An accessible atom is called *geometrically ergodic* if there exists $r_\alpha > 1$ such that
$$\sum_n r_\alpha^n |P^n(\boldsymbol{\alpha},\boldsymbol{\alpha}) - \pi(\boldsymbol{\alpha})| < \infty.$$
An accessible atom is called a *Kendall atom* of rate κ if there exists $\kappa > 1$ such that
$$U_\alpha^{(\kappa)}(\boldsymbol{\alpha},\boldsymbol{\alpha}) = \mathsf{E}_\alpha[\kappa^{\tau_\alpha}] < \infty.$$
Suppose that $f \geq 1$. An accessible atom is called *f-Kendall* of rate κ if there exists $\kappa > 1$ such that
$$\sup_{x\in\alpha} \mathsf{E}_x\Big[\sum_{n=0}^{\tau_\alpha - 1} f(\Phi_n)\kappa^n\Big] < \infty.$$

Equivalently, if f is bounded on the accessible atom $\boldsymbol{\alpha}$, then $\boldsymbol{\alpha}$ is f-Kendall of rate κ provided
$$U_\alpha^{(\kappa)}(\boldsymbol{\alpha}, f) = \mathsf{E}_\alpha\Big[\sum_{n=1}^{\tau_\alpha} f(\Phi_n)\kappa^n\Big] < \infty.$$
The application of Kendall's Theorem to chains admitting an atom comes from the following, which is straightforward from the assumption that $f \geq 1$, so that $U_\alpha^{(\kappa)}(\boldsymbol{\alpha}, f) \geq \mathsf{E}_\alpha[\kappa^{\tau_\alpha}]$.

Proposition 15.1.2 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible and aperiodic, and $\boldsymbol{\alpha}$ is an accessible Kendall atom. Then there exists $r_\alpha > 1$ and $R < \infty$ such that*
$$|P^n(\boldsymbol{\alpha},\boldsymbol{\alpha}) - \pi(\boldsymbol{\alpha})| \leq R r_\alpha^{-n}, \qquad n \to \infty.$$
$\square$

This enables us to control the first term in (15.11). To exploit the other bounds in (15.9)-(15.11) we also need to establish finiteness of the quantities $U_\alpha^{(\kappa)}(x, f)$ for values of x other than $\boldsymbol{\alpha}$.

Proposition 15.1.3 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible, and admits an f-Kendall atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$ of rate κ. Then the set*

$$S_f^\kappa := \{x : U_\alpha^{(\kappa)}(x, f) < \infty\} \tag{15.17}$$

is full and absorbing.

PROOF The kernel $U_\alpha^{(\kappa)}(x,\cdot)$ satisfies the identity

$$\int P(x, dy) U_\alpha^{(\kappa)}(y, B) = \kappa^{-1} U_\alpha^{(\kappa)}(x, B) + P(x, \boldsymbol{\alpha}) U_\alpha^{(\kappa)}(\boldsymbol{\alpha}, B)$$

and integrating against f gives

$$PU_\alpha^{(\kappa)}(x, f) = \kappa^{-1} U_\alpha^{(\kappa)}(x, f) + P(x, \boldsymbol{\alpha}) U_\alpha^{(\kappa)}(\boldsymbol{\alpha}, f).$$

Thus the set S_f^κ is absorbing, and since S_f^κ is non-empty it follows from Proposition 4.2.3 that S_f^κ is full. □

We now have sufficient structure to prove the geometric ergodic theorem when an atom exists with appropriate properties.

Theorem 15.1.4 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible, with invariant probability measure π, and that there exists an f-Kendall atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$ of rate κ.*

Then there exists a decomposition $\mathsf{X} = S^\kappa \cup N$ where S^κ is full and absorbing, such that for all $x \in S^\kappa$, some $R < \infty$, and some r with $r > 1$

$$\sum_n r^n \|P^n(x,\cdot) - \pi(\cdot)\|_f \le R U_\alpha^{(\kappa)}(x, f) < \infty. \tag{15.18}$$

PROOF By Proposition 15.1.3 the bounds (15.9) and (15.10), and the second term in the bound (15.11), are all finite for $x \in S^\kappa$; and Kendall's Theorem, as applied in Proposition 15.1.2, gives that for some $r_\alpha > 1$ the other term in (15.11) is also finite. The result follows with $r = \min(\kappa, r_\alpha)$. □

There is an alternative way of stating Theorem 15.1.4 in the simple geometric ergodicity case $f = 1$ which emphasizes the solidarity result in terms of ergodic properties rather than in terms of hitting time properties. The proof uses the same steps as the previous proof, and we omit it.

Theorem 15.1.5 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible, with invariant probability measure π, and that there is one geometrically ergodic atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$. Then there exists $\kappa > 1, r > 1$ and a decomposition $\mathsf{X} = S^\kappa \cup N$ where S^κ is full and absorbing, such that for some $R < \infty$ and all $x \in S^\kappa$*

$$\sum_n r^n \|P^n(x,\cdot) - \pi(\cdot)\| \le R\mathsf{E}_x[\kappa^{\tau_\alpha}] < \infty, \tag{15.19}$$

so that $\boldsymbol{\Phi}$ restricted to S^κ is also geometrically ergodic. □

15.1.4 Some geometrically ergodic chains on countable spaces

Forward recurrence time chains Consider as in Section 2.4 the forward recurrence time chain $\mathbf{V}^+$.

By construction, we have for this chain that

$$\mathsf{E}_1[r^{\tau_1}] = \sum_n r^n \mathsf{P}_1(\tau_1 = n) = \sum_n r^n p(n)$$

so that the chain is geometrically ergodic if and only if the distribution $p(n)$ has geometrically decreasing tails.

We will see, once we develop a drift criterion for geometric ergodicity, that this duality between geometric tails on increments and geometric rates of convergence to stationarity is repeated for many other models.

A non-geometrically ergodic example Not all ergodic chains on $\mathbb{Z}_+$ are geometrically ergodic, even if (as in the forward recurrence time chain) the steps to the right are geometrically decreasing. Consider a chain on $\mathbb{Z}_+$ with the transition matrix

$$\begin{array}{rcll} P(0,j) & = & \gamma_j, & j \in \mathbb{Z}_+ \\ P(j,j) & = & \beta_j, & j \in \mathbb{Z}_+ \\ P(j,0) & = & 1-\beta_j, & j \in \mathbb{Z}_+. \end{array} \tag{15.20}$$

where $\sum_j \gamma_j = 1$.

The mean return time from zero to itself is given by

$$\mathsf{E}_0[\tau_0] = \sum_j \gamma_j [1 + (1-\beta_j)^{-1}]$$

and the chain is thus ergodic if $\gamma_j > 0$ for all j (ensuring irreducibility and aperiodicity), and

$$\sum_j \gamma_j (1-\beta_j)^{-1} < \infty. \tag{15.21}$$

In this example

$$\mathsf{E}_0[r^{\tau_0}] \geq r \sum_j \gamma_j \mathsf{E}_j[r^{\tau_0}]$$

and

$$\mathsf{P}_j(\tau_0 > n) = \beta_j^n.$$

Hence if $\beta_j \to 1$ as $n \to \infty$, then the chain is not geometrically ergodic regardless of the structure of the distribution $\{\gamma_j\}$, even if $\gamma_n \to 0$ sufficiently fast to ensure that (15.21) holds.

Different rates of convergence Although it is possible to ensure a common rate of convergence in the Geometric Ergodic Theorem, there appears to be no simple way to ensure for a particular state that the rate is best possible. Indeed, in general this will not be the case.

To see this consider the matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix}$$

By direct inspection we find the diagonal elements have generating functions

$$\begin{array}{rcl} U^{(z)}(0,0) & = & 1 + z/4(1-z) \\ U^{(z)}(1,1) & = & 1 + z/2(1-z) + z/4(1-z) \\ U^{(z)}(2,2) & = & 1 + z/4(1-z) \end{array}$$

Thus the best rates for convergence of $P^n(0,0)$ and $P^n(2,2)$ to their limits $\pi(0) = \pi(2) = \frac{1}{4}$ are $\rho_0 = \rho_2 = 0$: the limits are indeed attained at every step. But the rate of convergence of $P^n(1,1)$ to $\pi(1) = \frac{1}{2}$ is at least $\rho_1 > \frac{1}{4}$.

The following more complex example shows that even on an arbitrarily large finite space $\{1, \ldots, N+1\}$ there may in fact be N different rates of convergence such that

$$|P^n(i,i) - \pi(i)| \leq M_i \rho_i^n.$$

Consider the matrix

$$P = \begin{bmatrix} \beta_1 & \alpha_1 & \alpha_1 & \ldots & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_1 & \beta_2 & \alpha_2 & \ldots & \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_2 & \beta_3 & \ldots & \alpha_3 & \alpha_3 & \alpha_3 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \beta_{N-1} & \alpha_{N-1} & \alpha_{N-1} \\ \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{N-1} & \beta_N & \alpha_N \\ \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{N-1} & \alpha_N & \beta_N \end{bmatrix}$$

so that

$$P(k,k) = \beta_k := 1 - \sum_1^{k-1} \alpha_j - (N+1-k)\alpha_k, \qquad 1 \leq k \leq N+1,$$

where the off-diagonal elements are ordered by

$$0 < \alpha_N < \alpha_{N-1} < \ldots < \alpha_2 < \alpha_1 \leq [N+1]^{-1}.$$

Since P is symmetric it is immediate that the invariant measure is given for all k by

$$\pi(k) = [N+1]^{-1}.$$

For this example it is possible to show [263] that the eigenvalues of P are distinct and are given by $\lambda_1 = 1$ and for $k = 2, \ldots, N+1$

$$\lambda_k = \beta_{N+2-k} - \alpha_{N+2-k}.$$

After considerable algebra it follows that for each k, there are positive constants $s(k,j)$ such that

$$P^m(k,k) - [N+1]^{-1} = \sum_{j=N+2-k}^{N+1} s(k,j)\lambda_j^m$$

and hence k has the exact "self-convergence" rate λ_{N+2-k}.

Moreover, $s(N+1,j) = s(N,j)$ for all $1 \leq j \leq N+1$, and so for the $N+1$ states there are N different "best" rates of convergence.

Thus our conclusion of a common rate parameter is the most that can be said.

15.2 Kendall sets and drift criteria

It is of course now obvious that we should try to move from the results valid for chains with atoms, to strongly aperiodic chains and thence to general aperiodic chains via the Nummelin splitting and the m-skeleton.

We first need to find conditions on the original chain under which the atom in the split chain is an f-Kendall atom. This will give the desired ergodic theorem for the split chain, which is then passed back to the original chain by exploiting a growth rate on the f-norm which holds for "f-geometrically regular chains". This extends the argument used in the proof of Lemma 14.3.2 to prove the f-Norm Ergodic Theorem in Chapter 14.

To do this we need to extend the concepts of Kendall atoms to general sets, and connect these with another and stronger drift condition: this has a dual purpose, for not only will it enable us to move relatively easily between chains, their skeletons, and their split forms, it will also give us a verifiable criterion for establishing geometric ergodicity.

15.2.1 f-Kendall sets and f-geometrically regular sets

The crucial aspect of a Kendall atom is that the return times to the atom from itself have a geometrically bounded distribution. There is an obvious extension of this idea to more general, non-atomic, sets.

Kendall sets and f-geometrically regular sets

A set $A \in \mathcal{B}(\mathsf{X})$ is called a *Kendall set* if there exists $\kappa > 1$ such that

$$\sup_{x \in A} \mathsf{E}_x[\kappa^{\tau_A}] < \infty.$$

A set $A \in \mathcal{B}(\mathsf{X})$ is called an *f-Kendall set* for a measurable $f : \mathsf{X} \to [1, \infty)$ if there exists $\kappa = \kappa(f) > 1$ such that

$$\sup_{x \in A} \mathsf{E}_x\Big[\sum_{k=0}^{\tau_A - 1} f(\Phi_k)\kappa^k\Big] < \infty. \tag{15.22}$$

A set $A \in \mathcal{B}(\mathsf{X})$ is called *f-geometrically regular* for a measurable $f : \mathsf{X} \to [1, \infty)$ if for each $B \in \mathcal{B}^+(\mathsf{X})$ there exists $r = r(f, B) > 1$ such that

$$\sup_{x \in A} \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B - 1} f(\Phi_k) r^k\Big] < \infty.$$

Clearly, since we have $r > 1$ in these definitions, an f-geometrically regular set is also f-regular. When a set or a chain is 1-geometrically regular then we will call it geometrically regular.

A Kendall set is, in an obvious way, "self-geometrically regular": return times to the set itself are geometrically bounded, although not necessarily hitting times on other sets.

As in (15.12), for any set C in $\mathcal{B}(\mathsf{X})$ the kernel $U_C^{(r)}(x,B)$ is given by

$$U_C^{(r)}(x,B) = \mathsf{E}_x\Big[\sum_{k=1}^{\tau_C} \mathbb{1}_B(\Phi_k) r^k\Big]; \tag{15.23}$$

this is again well defined for $r \geq 1$, although it may be infinite. We use this notation in our next result, which establishes that any petite f-Kendall set is actually f-geometrically regular. This is non-trivial to establish, and needs a somewhat delicate "geometric trials" argument.

Theorem 15.2.1 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible. Then the following are equivalent:*

(i) *The set $C \in \mathcal{B}(\mathsf{X})$ is a petite f-Kendall set.*

(ii) *The set C is f-geometrically regular and $C \in \mathcal{B}^+(\mathsf{X})$.*

PROOF To prove (ii)$\Rightarrow$(i) it is enough to show that A is petite, and this follows from Proposition 11.3.8, since a geometrically regular set is automatically regular.

To prove (i)$\Rightarrow$(ii) is considerably more difficult, although obviously since a Kendall set is Harris recurrent, it follows from Proposition 9.1.1 that any Kendall set is in $\mathcal{B}^+(\mathsf{X})$.

Suppose that C is an f-Kendall set of rate κ, let $1 < r \leq \kappa$, and define $U^{(r)}(x) = \mathsf{E}_x[r^{\tau_C}]$, so that $U^{(r)}$ is bounded on C. We set $M(r) = \sup_{x\in C} U^{(r)}(x) < \infty$. Put $\varepsilon = \log(r)/\log(\kappa)$: by Jensen's inequality,

$$M(r) = \sup_{x\in C} \mathsf{E}_x[\kappa^{\varepsilon\tau_C}] \leq M(\kappa)^{\varepsilon}.$$

From this bound we see that $M(r) \to 1$ as $r \downarrow 1$.

Let $\tau_C(n)$ denote the nth return time to the set C, where for convenience, we set $\tau_C(0) := 0$. We have by the strong Markov property and induction,

$$\begin{aligned}
\mathsf{E}_x[r^{\tau_C(n)}] &= \mathsf{E}_x[r^{\tau_C(n-1)+\theta^{\tau_C(n-1)}\tau_C}] \\
&= \mathsf{E}_x[r^{\tau_C(n-1)}\mathsf{E}_{\Phi_{\tau_C(n-1)}}[r^{\tau_C}]] \\
&\leq M(r)\,\mathsf{E}_x[r^{\tau_C(n-1)}] \\
&\leq (M(r))^{n-1}U^{(r)}(x), \qquad n \geq 1.
\end{aligned} \tag{15.24}$$

To prove the theorem we will combine this bound with the sample path bound, valid for any set $B \in \mathcal{B}(\mathsf{X})$,

$$\sum_{i=1}^{\tau_B} r^i f(\Phi_i) \leq \sum_{n=0}^{\infty}\Big(\sum_{j=\tau_C(n)+1}^{\tau_C(n+1)} r^j f(\Phi_j)\Big)\mathbb{1}\{\tau_B > \tau_C(n)\}.$$

Taking expectations and applying the strong Markov property gives

$$
\begin{aligned}
U_B^{(r)}(x,f) &\le \sum_{n=0}^{\infty} \mathsf{E}_x\Big[\mathbb{1}\{\tau_B > \tau_C(n)\} r^{\tau_C(n)} \mathsf{E}_{\Phi_{\tau_C(n)}}\Big[\sum_{j=1}^{\tau_C} r^j f(\Phi_j)\Big]\Big] \\
&\le \sup_{x\in C} U_C^{(r)}(x,f) \sum_{n=0}^{\infty} \mathsf{E}_x\Big[\mathbb{1}\{\tau_B > \tau_C(n)\} r^{\tau_C(n)}\Big]. \qquad (15.25)
\end{aligned}
$$

For any $0 < \gamma < 1$, $n \ge 0$, and positive numbers x and y we have the bound $xy \le \gamma^n x^2 + \gamma^{-n} y^2$. Applying this bound with $x = r^{\tau_C(n)}$ and $y = \mathbb{1}\{\tau_C(n) < \tau_B\}$ in (15.25), and setting $M_f(r) = \sup_{x\in C} U_C^{(r)}(x,f)$ we obtain for any $B \in \mathcal{B}(\mathsf{X})$,

$$
\begin{aligned}
U_B^{(r)}(x,f) &\le M_f(r) \sum_{n=0}^{\infty} \Big\{\gamma^n \mathsf{E}_x[r^{2\tau_C(n)}] + \gamma^{-n}\mathsf{E}_x[\mathbb{1}\{\tau_C(n) < \tau_B\}]\Big\} \\
&\le M_f(r)\Big\{\sum_{n=0}^{\infty} \gamma^n (M(r^2))^n U^{(r^2)}(x) \\
&\qquad + \sum_{n=0}^{\infty} \gamma^{-n} \mathsf{P}_x\{\tau_C(n) < \tau_B\}\Big\}, \qquad (15.26)
\end{aligned}
$$

where we have used (15.24). We still need to prove the right hand side of (15.26) is finite. Suppose now that for some $R < \infty$, $\rho < 1$, and any $x \in \mathsf{X}$,

$$
\mathsf{P}_x\{\tau_C(n) < \tau_B\} \le R\rho^n. \qquad (15.27)
$$

Choosing $\rho < \gamma < 1$ in (15.26) gives

$$
U_B^{(r)}(x,f) \le M_f(r)\Big\{U^{(r^2)}(x)\sum_{n=0}^{\infty}(\gamma M(r^2))^n + \frac{R}{1-\gamma^{-1}\rho}\Big\}.
$$

With γ so fixed, we can now choose $r > 1$ so close to unity that $\gamma M(r^2) < 1$ to obtain

$$
U_B^{(r)}(x,f) \le M_f(r)\Big\{\frac{U^{(r^2)}(x)}{1-\gamma M(r^2)} + \frac{R}{1-\gamma^{-1}\rho}\Big\}.
$$

and the result holds.

To complete the proof, it is thus enough to bound $\mathsf{P}_x\{\tau_C(n) < \tau_B\}$ by a geometric series as in (15.27). Since C is petite, there exists $n_0 \in \mathbb{Z}_+$, $c < 1$, such that

$$
\mathsf{P}_x\{\tau_C(n_0) < \tau_B\} \le \mathsf{P}_x\{n_0 < \tau_B\} \le c, \qquad x \in C,
$$

and by the strong Markov property it follows that with $m_0 = n_0 + 1$,

$$
\mathsf{P}_x\{\tau_C(m_0) < \tau_B\} \le c, \qquad x \in \mathsf{X}.
$$

Hence, using the identity

$$
\mathbb{1}\{\tau_C(mm_0) < \tau_B\} = \mathbb{1}\{\tau_C([m-1]m_0) < \tau_B\}\theta^{\tau_C([m-1]m_0)}\mathbb{1}\{\tau_C(m_0) < \tau_B\}
$$

we have again by the strong Markov property that for all $x \in \mathsf{X}$, $m \ge 1$,

$$
\begin{aligned}
\mathsf{P}_x\{\tau_C(mm_0) < \tau_B\} &= \mathsf{E}_x\Big\{\mathbb{1}\{\tau_C([m-1]m_0) < \tau_B\}\mathsf{P}_{\Phi_{\tau_C([m-1]m_0)}}\{\tau_C(m_0) < \tau_B\}\Big\} \\
&\le c\mathsf{P}_x\{\tau_C([m-1]m_0) < \tau_B\} \\
&\le c^m
\end{aligned}
$$

and it now follows easily that (15.27) holds. □

Notice specifically in this result that there may be a separate rate of convergence r for each of the quantities

$$\sup_{x \in C} U_B^{(r)}(x, f)$$

depending on the quantity ρ in (15.27): intuitively, for a set B "far away" from C it may take many visits to C before an excursion reaches B, and so the value of r will be correspondingly closer to unity.

15.2.2 The geometric drift condition

Whilst for strongly aperiodic chains an approach to geometric ergodicity is possible with the tools we now have directly through petite sets, in order to move from strongly aperiodic to aperiodic chains through skeleton chains and splitting methods an attractive theoretical route is through another set of drift inequalities.

This has, as usual, the enormous practical benefit of providing a set of verifiable conditions for geometric ergodicity. The drift condition appropriate for geometric convergence is:

Geometric Drift Towards C

(V4) There exists an extended-real valued function $V : \mathsf{X} \to [1, \infty]$, a measurable set C, and constants $\beta > 0$, $b < \infty$,

$$\Delta V(x) \le -\beta V(x) + b \mathbb{1}_C(x), \quad x \in \mathsf{X}. \tag{15.28}$$

We see at once that (V4) is just (V3) in the special case where $f = \beta V$. From this observation we can borrow several results from the previous chapter, and use the approach there as a guide.

We first spell out some useful properties of solutions to the drift inequality in (15.28), analogous to those we found for (14.16).

Lemma 15.2.2 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible.*

(i) *If V satisfies (15.28) then $\{V < \infty\}$ is either empty or absorbing and full.*

(ii) *If (15.28) holds for a petite set C then V is unbounded off petite sets.*

PROOF Since (15.28) implies $PV \le V + b$ the set $\{V < \infty\}$ is absorbing; hence if it is non-empty it is full, by Proposition 4.2.3.

Since $V \ge 1$, we see that (V4) implies that (V2) holds with $V' = V/(1-\beta)$. From Lemma 11.3.7 it then follows that V' (and hence obviously V) is unbounded off petite sets. □

We now begin a more detailed evaluation of the consequences of (V4). We first give a probabilistic form for one solution to the drift condition (V4), which will prove that (15.2) implies (15.3) has a solution.

Using the kernel $U_C^{(r)}$ we define a further kernel $G_C^{(r)}$ as $G_C^{(r)} = I + I_{C^c} U_C^{(r)}$. For any $x \in \mathsf{X}$, $B \in \mathcal{B}(\mathsf{X})$, this has the interpretation

$$G_C^{(r)}(x, B) = \mathsf{E}_x \Big[\sum_{k=0}^{\sigma_C} \mathbb{1}_B(\Phi_k) r^k \Big]. \tag{15.29}$$

The kernel $G_C^{(r)}(x, B)$ gives us the solution we seek to (15.28).

Lemma 15.2.3 *Suppose that $C \in \mathcal{B}(\mathsf{X})$, and let $r > 1$. Then the kernel $G_C^{(r)}$ satisfies*

$$PG_C^{(r)} = r^{-1} G_C^{(r)} - r^{-1} I + r^{-1} I_C U_C^{(r)}$$

so that in particular for $\beta = 1 - r^{-1}$

$$PG_C^{(r)} - G_C^{(r)} = \Delta G_C^{(r)} \le -\beta G_C^{(r)} + r^{-1} I_C U_C^{(r)}. \tag{15.30}$$

PROOF The kernel $U_C^{(r)}$ satisfies the simple identity

$$U_C^{(r)} = rP + rPI_{C^c} U_C^{(r)}. \tag{15.31}$$

Hence the kernel $G_C^{(r)}$ satisfies the chain of identities

$$PG_C^{(r)} = P + PI_{C^c} U_C^{(r)} = r^{-1} U_C^{(r)} = r^{-1}[G_C^{(r)} - I + I_C U_C^{(r)}].$$

□

This now gives us the easier direction of the duality between the existence of f-Kendall sets and solutions to (15.28).

Theorem 15.2.4 *Suppose that Φ is ψ-irreducible, and admits an f-Kendall set $C \in \mathcal{B}^+(\mathsf{X})$ for some $f \ge 1$. Then the function $V(x) = G_C^{(\kappa)}(x, f) \ge f(x)$ is a solution to (V4).*

PROOF We have from (15.30) that, by the f-Kendall property, for some $M < \infty$ and $r > 1$,

$$\Delta V \le -\beta V + r^{-1} M \mathbb{1}_C$$

and so the function V satisfies (V4). □

15.2.3 Other solutions of the drift inequalities

We have shown that the existence of f-geometrically regular sets will lead to solutions of (V4). We now show that the converse also holds.

The tool we need in order to consider properties of general solutions to (15.28) is the following "geometric" generalization of the Comparison Theorem.

Theorem 15.2.5 *If (V4) holds then for any $r \in (1,(1-\beta)^{-1})$ there exists $\varepsilon > 0$ such that for any first entrance time τ_B,*

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} V(\Phi_k)r^k\Big] \le \varepsilon^{-1}r^{-1}V(x) + \varepsilon^{-1}b\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} \mathbb{1}_C(\Phi_k)r^k\Big]$$

and hence in particular choosing $B = C$

$$V(x) \le \mathsf{E}_x\Big[\sum_{k=0}^{\tau_C-1} V(\Phi_k)r^k\Big] \le \varepsilon^{-1}r^{-1}V(x) + \varepsilon^{-1}b\mathbb{1}_C(x). \tag{15.32}$$

Proof We have the bound

$$PV \le r^{-1}V - \varepsilon V + b\mathbb{1}_C$$

where $0 < \varepsilon < \beta$ is the solution to $r = (1-\beta+\varepsilon)^{-1}$. Defining

$$Z_k = r^k V(\Phi_k)$$

for $k \in \mathbb{Z}_+$, it follows that

$$\begin{aligned}
\mathsf{E}[Z_{k+1} \mid \mathcal{F}_k^{\Phi}] &= r^{k+1}\mathsf{E}[V(\Phi_{k+1}) \mid \mathcal{F}_k^{\Phi}] \\
&\le r^{k+1}\{r^{-1}V(\Phi_k) - \varepsilon V(\Phi_k) + b\mathbb{1}_C(\Phi_k)\} \\
&= Z_k - \varepsilon r^{k+1}V(\Phi_k) + r^{k+1}b\mathbb{1}_C(\Phi_k).
\end{aligned}$$

Choosing $f_k(x) = \varepsilon r^{k+1}V(x)$ and $s_k(x) = br^{k+1}\mathbb{1}_C(x)$, we have by Proposition 11.3.2

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} \varepsilon r^{k+1}V(\Phi_k)\Big] \le Z_0(x) + \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} r^{k+1}b\mathbb{1}_C(\Phi_k)\Big].$$

Multiplying through by $\varepsilon^{-1}r^{-1}$ and noting that $Z_0(x) = V(x)$, we obtain the required bound.

The particular form with $B = C$ is then straightforward. □

We use this result to prove that in general, sublevel sets of solutions V to (15.28) are V-geometrically regular.

Theorem 15.2.6 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible, and that (V4) holds for a function V and a petite set C.*

If V is bounded on $A \in \mathcal{B}(\mathsf{X})$, then A is V-geometrically regular.

Proof We first show that if V is bounded on A, then $A \subseteq D$ where D is a V-Kendall set.

Assume (V4) holds, let $\rho = 1 - \beta$, and fix $\rho < r^{-1} < 1$. Now consider the set D defined by

$$D := \Big\{x : V(x) \leq \frac{M+b}{r^{-1}-\rho}\Big\}, \tag{15.33}$$

where the integer $M > 0$ is chosen so that $A \subseteq D$ (which is possible because the function V is bounded on A) and $D \in \mathcal{B}^+(\mathsf{X})$, which must be the case for sufficiently large M from Lemma 15.2.2 (i).

Using (V4) we have

$$\begin{aligned} PV(x) &\leq r^{-1}V(x) - (r^{-1}-\rho)V(x) + b\mathbb{1}_C(x) \\ &\leq r^{-1}V(x) - M, \qquad x \in D^c. \end{aligned}$$

Since $PV(x) \leq V(x) + b$, which is bounded on D, it follows that

$$PV \leq r^{-1}V + c\mathbb{1}_D$$

for some $c < \infty$. Thus we have shown that (V4) holds with D in place of C.

Hence using (15.32) there exists $s > 1$ and $\varepsilon > 0$ such that

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_D - 1} s^k V(\Phi_k)\Big] \quad \leq \quad \varepsilon^{-1}s^{-1}V(x) + \varepsilon^{-1}c\mathbb{1}_D(x). \tag{15.34}$$

Since V is bounded on D by construction, this shows that D is V-Kendall as required.

By Lemma 15.2.2 (ii) the function V is unbounded off petite sets, and therefore the set D is petite. Applying Theorem 15.2.1 we see that D is V-geometrically regular.

Finally, since by definition any subset of a V-geometrically regular set is itself V-geometrically regular, we have that A inherits this property from D. □

As a simple consequence of Theorem 15.2.6 we can construct, given just one f-Kendall set in $\mathcal{B}^+(\mathsf{X})$, an increasing sequence of f-geometrically regular sets whose union is full: indeed we have a somewhat more detailed description than this.

Theorem 15.2.7 *If there exists an f-Kendall set $C \in \mathcal{B}^+(\mathsf{X})$, then there exists $V \geq f$ and an increasing sequence $\{C_V(i) : i \in \mathbb{Z}_+\}$ of V-geometrically regular sets whose union is full.*

Proof Let $V(x) = G_C^{(r)}(x, f)$. Then V satisfies (V4) and by Theorem 15.2.6 the set $C_V(n) := \{x : V(x) \leq n\}$ is V-geometrically regular for each n. Since $S_V = \{V < \infty\}$ is a full absorbing subset of X, the result follows. □

The following alternative form of (V4) will simplify some of the calculations performed later.

Lemma 15.2.8 *The drift condition (V4) holds with a petite set C if and only if V is unbounded off petite sets and*

$$PV \leq \lambda V + L \tag{15.35}$$

for some $\lambda < 1$, $L < \infty$.

PROOF If (V4) holds, then (15.35) immediately follows. Lemma 15.2.2 states that the function V is unbounded off petite sets.

Conversely, if (15.35) holds for a function V which is unbounded off petite sets then set $\beta = \frac{1}{2}(1-\lambda)$ and define the petite set C as

$$C = \{x \in \mathsf{X} : V(x) \le L/\beta\}$$

It follows that $\Delta V \le -\beta V + L\mathbb{1}_C$ so that (V4) is satisfied. □

We will find in several examples on topological spaces that the bound (15.35) is obtained for some norm-like function V and compact C. If the Markov chain is a ψ-irreducible T-chain it follows from Lemma 15.2.8 that (V4) holds and then that the chain is V-geometrically ergodic.

Although the result that one can use the same function V in both sides of

$$\sum_n r^n \|P^n(x,\,\cdot\,) - \pi\|_V \le RV(x).$$

is an important one, it also has one drawback: as we have larger functions on the left, the bounds on the distance to $\pi(V)$ also increase.

Overall it is not clear when one can have a best common bound on the distance $\|P^n(x,\,\cdot\,) - \pi\|_V$ independent of V; indeed, the example in Section 16.2.2 shows that as V increases then one might even lose the geometric nature of the convergence.

However, the following result shows that one can obtain a smaller x-dependent bound in the Geometric Ergodic Theorem if one is willing to use a smaller function V in the application of the V-norm.

Lemma 15.2.9 *If (V4) holds for V, and some petite set C, then (V4) also holds for the function $\sqrt{V}$ and some petite set C.*

PROOF If (V4) holds for the finite-valued function V then by Lemma 15.2.8 V is unbounded off petite sets and (15.35) holds for some $\lambda < 1$ and $L < \infty$. Letting $V'(x) = \sqrt{V(x)}$, $x \in \mathsf{X}$, we have by Jensen's inequality,

$$\begin{aligned} PV'(x) \le \sqrt{PV(x)} &\le \sqrt{\lambda V + L} \\ &\le \sqrt{\lambda}\sqrt{V} + \frac{L}{2\sqrt{\lambda}} \qquad \text{since } V \ge 1 \\ &= \sqrt{\lambda}V' + \frac{L}{2\sqrt{\lambda}}, \end{aligned}$$

which together with Lemma 15.2.8 implies that (V4) holds with V replaced by $\sqrt{V}$. □

15.3 f-Geometric regularity of Φ and Φ^n

15.3.1 f-Geometric regularity of chains

There are two aspects to the f-geometric regularity of sets that we need in moving to our prime purpose in this chapter, namely proving the f-geometric convergence part of the Geometric Ergodic Theorem.

The first is to locate sets from which the hitting times on other sets are geometrically fast. For the purpose of our convergence theorems, we need this in a specific way: from an f-Kendall set we will only need to show that the hitting times on a split atom are geometrically fast, and in effect this merely requires that hitting times on a (rather specific) subset of a petite set be geometrically fast. Indeed, note that in the case with an atom we only needed the f-Kendall (or self f-geometric regularity) property of the atom, and there was no need to prove that the atom was fully f-geometrically regular. The other structural results shown in the previous section are an unexpectedly rich by-product of the requirement to delineate the geometric bounds on subsets of petite sets. This approach also gives, as a more directly useful outcome, an approach to working with the m-skeleton from which we will deduce rates of convergence.

Secondly, we can see from the Regenerative Decomposition that we will need the analogue of Proposition 15.1.3: that is, we need to ensure that for some specific set there is a fixed geometric bound on the hitting times of the set from arbitrary starting points. This motivates the next definition.

f-Geometric Regularity of $\boldsymbol{\Phi}$

The chain $\boldsymbol{\Phi}$ is called f-geometrically regular if there exists a petite set C and a fixed constant $\kappa > 1$ such that

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_C - 1} f(\Phi_k)\kappa^k\Big] \tag{15.36}$$

is finite for all $x \in \mathsf{X}$ and bounded on C.

Observe that when κ is taken equal to one, this definition then becomes f-regularity, whilst the boundedness on C implies f-geometric regularity of the set C from Theorem 15.2.1: it is the finiteness from arbitrary initial points that is new in this definition.

The following consequence of f-regularity follows immediately from the strong Markov property and f-geometric regularity of the set C used in (15.36).

Proposition 15.3.1 *If $\boldsymbol{\Phi}$ is f-geometrically regular so that (15.36) holds for a petite set C then for each $B \in \mathcal{B}^+(\mathsf{X})$ there exists $r = r(B) > 1$ and $c(B) < \infty$ such that*

$$U_B^{(r)}(x, f) \le c(B) U_C^{(r)}(x, f). \tag{15.37}$$

$\square$

By now the techniques we have developed ensure that f-geometrically regularity is relatively easy to verify.

Proposition 15.3.2 *If there is one petite f-Kendall set C then there is a decomposition*

$$\mathsf{X} = S_f \cup N$$

where S_f is full and absorbing, and Φ restricted to S_f is f-geometrically regular.

PROOF We know from Theorem 15.2.1 that when a petite f-Kendall set C exists then C is V-geometrically regular, where $V(x) = G_C^{(r)}(x, f)$ for some $r > 1$. Since V then satisfies (V4) from Lemma 15.2.3, it follows from Lemma 15.2.2 that $S_f = \{V < \infty\}$ is absorbing and full. Now as in (15.32) we have for some $\kappa > 1$

$$V(x) \le \mathsf{E}_x\Big[\sum_{n=0}^{\tau_C - 1} V(\Phi_n)\kappa^n\Big] \le \varepsilon^{-1}\kappa^{-1}V(x) + \varepsilon^{-1}c\mathbb{1}_C(x) \tag{15.38}$$

and since the right hand side is finite on S_f the chain restricted to S_f is V-geometrically regular, and hence also f-geometrically regular since $f \le V$. □

The existence of an everywhere finite solution to the drift inequality (V4) is equivalent to f-geometric regularity, imitating the similar characterization of f-regularity. We have

Theorem 15.3.3 *Suppose that (V4) holds for a petite set C and a function V which is everywhere finite. Then Φ is V-geometrically regular, and for each $B \in \mathcal{B}^+(\mathsf{X})$ there exists $c(B) < \infty$ such that*

$$U_B^{(r)}(x, V) \le c(B)V(x).$$

Conversely, if Φ is f-geometrically regular, then there exists a petite set C and a function $V \ge f$ which is everywhere finite and which satisfies (V4).

PROOF Suppose that (V4) holds with V everywhere finite and C petite. As in the proof of Theorem 15.2.6, there exists a petite set D on which V is bounded, and as in (15.34) there is then $r > 1$ and a constant d such that

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_D - 1} V(\Phi_k)r^k\Big] \le dV(x).$$

Hence Φ is V-geometrically regular, and the required bound follows from Proposition 15.3.1.

For the converse, take $V(x) = G_C^{(r)}(x, f)$ where C is the petite set used in the definition of f-geometric regularity. □

This approach, using solutions V to (V4) to bound (15.36), is in effect an extended version of the method used in the atomic case to prove Proposition 15.1.3.

15.3.2 Connections between Φ and Φ^n

A striking consequence of the characterization of geometric regularity in terms of the solution of (V4) is that we can prove almost instantly that if a set C is f-geometrically regular, and if Φ is aperiodic, then C is also f-geometrically regular for every skeleton chain.

Theorem 15.3.4 *Suppose that Φ is ψ-irreducible and aperiodic.*

(i) *If V satisfies (V4) with a petite set C then for any n-skeleton, the function V also satisfies (V4) for some set C' which is petite for the n-skeleton.*

(ii) *If C is f-geometrically regular then C is f-geometrically regular for the chain Φ^n for any $n \geq 1$.*

Proof **(i)** Suppose $\rho = 1 - \beta$ and $0 < \varepsilon < \rho - \rho^n$. By iteration we have using Lemma 14.2.8 that for some petite set C',

$$P^n V \leq \rho^n V + b \sum_{i=0}^{n-1} P^i \mathbb{1}_C \leq \rho^n V + bm\mathbb{1}_{C'} + \varepsilon.$$

Since $V \geq 1$ this gives

$$P^n V \leq \rho V + bm\mathbb{1}_{C'}, \tag{15.39}$$

and hence (V4) holds for the n-skeleton.

(ii) If C is f-geometrically regular then we know that (V4) holds with $V = G_C^{(r)}(x, f)$. We can then apply Theorem 15.2.6 to the n-skeleton and the result follows. □

Given this together with Theorem 15.3.3, which characterizes f-geometric regularity, the following result is obvious:

Theorem 15.3.5 *If Φ is f-geometrically regular and aperiodic, then every skeleton is also f-geometrically regular.* □

We round out this series of equivalences by showing not only that the skeletons inherit f-geometric regularity properties from the chain, but that we can go in the other direction also.

Recall from (14.22) that for any positive function g on X, we write $g^{(m)} = \sum_{i=0}^{m-1} P^i g$. Then we have, as a geometric analogue of Theorem 14.2.9,

Theorem 15.3.6 *Suppose that Φ is ψ-irreducible and aperiodic. Then $C \in \mathcal{B}^+(\mathsf{X})$ is f-geometrically regular if and only if it is $f^{(m)}$-geometrically regular for any one, and then every, m-skeleton chain.*

Proof Letting τ_B^m denote the hitting time for the skeleton, we have by the Markov property, for any $B \in \mathcal{B}^+(\mathsf{X})$ and $r > 1$,

$$\begin{aligned}
\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B^m - 1} r^{km} \sum_{i=0}^{m-1} P^i f(\Phi_{km})\Big] &\geq r^{-m}\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B^m - 1} \sum_{i=0}^{m-1} r^{km+i} f(\Phi_{km+i})\Big] \\
&\geq r^{-m}\mathsf{E}_x\Big[\sum_{j=0}^{\tau_B - 1} r^j f(\Phi_j)\Big].
\end{aligned}$$

If C is $f^{(m)}$-geometrically regular for an m-skeleton then the left hand side is bounded over C for some $r > 1$ and hence the set C is also f-geometrically regular.

Conversely, if $C \in \mathcal{B}^+(\mathsf{X})$ is f-geometrically regular then it follows from Theorem 15.2.4 that (V4) holds for a function $V \geq f$ which is bounded on C.

Thus we have from (15.39) and a further application of Lemma 14.2.8 that for some petite set C'' and $\rho' < 1$

$$P^m V^{(m)} \le \rho V^{(m)} + mb\mathbb{1}_{C'}^{(m)} \le \rho' V^{(m)} + mb\mathbb{1}_{C''}.$$

and thus (V4) holds for the m-skeleton. Since $V^{(m)}$ is bounded on C by (15.39), we have from Theorem 15.3.3 that C is $V^{(m)}$-geometrically regular for the m-skeleton. □

This gives the following solidarity result.

Theorem 15.3.7 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible and aperiodic. Then $\boldsymbol{\Phi}$ is f-geometrically regular if and only if each m-skeleton is $f^{(m)}$-geometrically regular.* □

15.4 f-Geometric ergodicity for general chains

We now have the results that we need to prove the geometrically ergodic limit (15.4). Using the result in Section 15.1.3 for a chain possessing an atom we immediately obtain the desired ergodic theorem for strongly aperiodic chains. We then consider the m-skeleton chain: we have proved that when $\boldsymbol{\Phi}$ is f-geometrically regular then so is each m-skeleton. For aperiodic chains, there always exists some $m \ge 1$ such that the m-skeleton is strongly aperiodic, and hence as in Chapter 14 we can prove geometric ergodicity using this strongly aperiodic skeleton chain.

We follow these steps in the proof of the following theorem.

Theorem 15.4.1 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible and aperiodic, and that there is one f-Kendall petite set $C \in \mathcal{B}(\mathsf{X})$.*

Then there exists $\kappa > 1$ and an absorbing full set S_f^κ on which

$$\mathsf{E}_x[\sum_{k=0}^{\tau_C - 1} f(\Phi_k)\kappa^k]$$

is finite, and for all $x \in S_f^\kappa$,

$$\sum_n r^n \, \|P^n(x,\,\cdot\,) - \pi\|_f \le R\,\mathsf{E}_x[\sum_{k=0}^{\tau_C} f(\Phi_k)\kappa^k]$$

for some $r > 1$ and $R < \infty$ independent of x.

Proof This proof is in several steps, from the atomic through the strongly aperiodic to the general aperiodic case. In all cases we use the fact that the seemingly relatively weak f-Kendall petite assumption on C implies that C is f-geometrically regular and in $\mathcal{B}^+(\mathsf{X})$ from Theorem 15.2.1.

Under the conditions of the theorem it follows from Theorem 15.2.4 that

$$V(x) = \mathsf{E}_x\Big[\sum_{k=0}^{\sigma_C} f(\Phi_k)\kappa^k\Big] \ge f(x) \tag{15.40}$$

is a solution to (V4) which is bounded on the set C, and the set $S_f^\kappa = \{x : V(x) < \infty\}$ is absorbing, full, and contains the set C. This will turn out to be the set required for the result.

(i) Suppose first that the set C contains an accessible atom $\boldsymbol{\alpha}$. We know then that the result is true from Theorem 15.1.4, with the bound on the f-norm convergence given from (15.18) and (15.37) by

$$\mathsf{E}_x[\sum_{k=0}^{\tau_\alpha-1} f(\Phi_k)\kappa^k] \le c(\boldsymbol{\alpha})\mathsf{E}_x[\sum_{k=0}^{\tau_C-1} f(\Phi_k)\kappa^k]$$

for some $\kappa > 1$ and a constant $c(\boldsymbol{\alpha}) < \infty$.

(ii) Consider next the case where the chain is strongly aperiodic, and this time assume that $C \in \mathcal{B}^+(\mathsf{X})$ is a ν_1-small set with $\nu_1(C^c) = 0$. Clearly this will not always be the case, but in part (iii) of the proof we see that this is no loss in generality.

To prove the theorem we abandon the function f and prove V-geometric ergodicity for the chain restricted to S_f^κ and the function (15.40). By Theorem 15.3.3 applied to the chain restricted to S_f^κ we have that for some constants $c < \infty$, $r > 1$,

$$\mathsf{E}_x\Big[\sum_{k=1}^{\tau_C} V(\Phi_k)r^k\Big] \le cV(x). \tag{15.41}$$

Now consider the chain split on C. Exactly as in the proof of Proposition 14.3.1 we have that

$$\check{\mathsf{E}}_{x_i}\Big[\sum_{k=1}^{\tau_{C_0\cup C_1}} \check{V}(\check{\Phi}_k)r^k\Big] \le c'\check{V}(x_i)$$

where $c' \ge c$ and $\check{V}$ is defined on $\check{\mathsf{X}}$ by $\check{V}(x_i) = V(x)$, $x \in \mathsf{X}$, $i = 0, 1$.

But this implies that $\check{\boldsymbol{\alpha}}$ is a $\check{V}$-Kendall atom, and so from step (i) above we see that for some $r_0 > 1$, $c'' < \infty$,

$$\sum_n r_0^n \|\check{P}^n(x_i, \cdot\,) - \check{\pi}\|_{\check{V}} \le c''\check{V}(x_i)$$

for all $x_i \in (S_f^\kappa)_0 \cup \mathsf{X}_1$.

It is then immediate that the original (unsplit) chain restricted to S_f^κ is V-geometrically ergodic and that

$$\sum_n r_0^n \,\|P^n(x, \cdot\,) - \pi\|_V \le c''V(x)$$

From the definition of V and the bound $V \ge f$ this proves the theorem when C is ν_1-small.

(iii) Now let us move to the general aperiodic case. Choose m so that the set C is itself ν_m-small with $\nu_m(C^c) = 0$: we know that this is possible from Theorem 5.5.7.

By Theorem 15.3.3 and Theorem 15.3.5 the chain and the m-skeleton restricted to S_f^κ are both V-geometrically regular. Moreover, by Theorem 15.3.3 and Theorem 15.3.4 we have for some constants $d < \infty$, $r > 1$,

$$\mathsf{E}_x\Big[\sum_{k=1}^{\tau_C^m} V(\Phi_k)r^k\Big] \le dV(x) \tag{15.42}$$

where as usual τ_C^m denotes the hitting time for the m-skeleton. From (ii), since m is chosen specifically so that C is "ν_1-small" for the m-skeleton, there exists $c < \infty$ with

$$\|P^{nm}(x, \cdot\,) - \pi\|_V \le cV(x)r_0^{-n}, \qquad n \in \mathbb{Z}_+,\ x \in S_f^\kappa.$$

We now need to compare this term with the convergence of the one-step transition probabilities, and we do not have the contraction property of the total variation norm available to do this. But if (V4) holds for V then we have that

$$PV(x) \leq V(x) + b \leq (1+b)V(x),$$

and hence for any $g \leq V$,

$$\begin{aligned} |P^{n+1}(x,g) - \pi(g)| &= |P^n(x,Pg) - \pi(Pg)| \\ &\leq \|P^n(x,\,\cdot\,) - \pi\|_{(1+b)V} \\ &= (1+b)\|P^n(x,\,\cdot\,) - \pi\|_V. \end{aligned}$$

Thus we have the bound

$$\|P^{n+1}(x,\,\cdot\,) - \pi\|_V \leq (1+b)\|P^n(x,\,\cdot\,) - \pi\|_V. \tag{15.43}$$

Now observe that for any $k \in \mathbb{Z}_+$, if we write $k = nm + i$ with $0 \leq i \leq m-1$, we obtain from (15.43) the bound, for any $x \in S_f^\kappa$

$$\begin{aligned} \|P^k(x,\,\cdot\,) - \pi\|_V &\leq (1+b)^m \|P^{nm}(x,\,\cdot\,) - \pi\|_V \\ &\leq (1+b)^m cV(x) r_0^{-n} \\ &\leq (1+b)^m c r_0 V(x) (r_0^{1/m})^{-k}, \end{aligned}$$

and the theorem is proved. □

Intuitively it seems obvious from the method of proof we have used here that f-geometric ergodicity will imply f-geometric regularity for any f, but of course the inequalities in the Regenerative Decomposition are all in one direction, and so we need to be careful in proving this result.

Theorem 15.4.2 *If $\boldsymbol{\Phi}$ is f-geometrically ergodic then there is a full absorbing set S such that $\boldsymbol{\Phi}$ is f-geometrically regular when restricted to S.*

Proof Let us first assume there is an accessible atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$, and that $r > 1$ is such that

$$\sum_n r^n \|P^n(\boldsymbol{\alpha},\,\cdot\,) - \pi\|_f < \infty.$$

Using the last exit decomposition (8.21) over the times of entry to $\boldsymbol{\alpha}$, we have as in the Regenerative Decomposition (13.48)

$$P^n(\boldsymbol{\alpha}, f) - \pi(f) \geq (u - \pi(\boldsymbol{\alpha})) * t_f(n) + \pi(\boldsymbol{\alpha}) \sum_{j=n+1}^{\infty} t_f(j). \tag{15.44}$$

Multiplying by r^n and summing both sides of (15.44) would seem to indicate that $\boldsymbol{\alpha}$ is an f-Kendall atom of rate r, save for the fact that the first term may be negative, so that we could have both positive and negative infinite terms in this sum in principle. We need a little more delicate argument to get around this.

By truncating the last term and then multiplying by $s^n, s \leq r$ and summing to N, we do have

$$\sum_{n=0}^N s^n(P^n(\boldsymbol{\alpha},f)-\pi(f)) \geq [\sum_{n=0}^N s^n t_f(n)[\sum_{k=0}^{N-n} s^k(u(k)-\pi(\boldsymbol{\alpha}))]] \\ +\pi(\boldsymbol{\alpha})\sum_{n=0}^N s^n \sum_{j=n+1}^N t_f(j). \tag{15.45}$$

Let us write $c_N(f,s)=\sum_{n=0}^N s^n t_f(n)$, and $d(s)=\sum_{n=0}^\infty s^n|u(n)-\pi(\boldsymbol{\alpha})|$. We can bound the first term in (15.45) in absolute value by $d(s)c_N(f,s)$, so in particular as $s\downarrow 1$, by monotonicity of $d(s)$ we know that the middle term is no more negative than $-d(r)c_N(f,s)$.

On the other hand, the third term is by Fubini's Theorem given by

$$\pi(\boldsymbol{\alpha})[s-1]^{-1}\sum_{n=0}^N t_f(n)(s^n-1) \geq [s-1]^{-1}[\pi(\boldsymbol{\alpha})c_N(f,s)-\pi(f)-\pi(\boldsymbol{\alpha})f(\boldsymbol{\alpha})]. \tag{15.46}$$

Suppose now that $\boldsymbol{\alpha}$ is not f-Kendall. Then for any $s>1$ we have that $c_N(f,s)$ is unbounded as N becomes large. Fix s sufficiently small that $\pi(\boldsymbol{\alpha})[s-1]^{-1}>d(r)$; then we have that the right hand side of (15.45) is greater than

$$c_N(f,s)[\pi(\boldsymbol{\alpha})[s-1]^{-1}-d(r)]-(\pi(f)+\pi(\boldsymbol{\alpha})f(\boldsymbol{\alpha}))/(1-s)$$

which tends to infinity as $N\to\infty$. This clearly contradicts the finiteness of the left side of (15.45). Consequently $\boldsymbol{\alpha}$ is f-Kendall of rate s for some $s<r$, and then the chain is f-geometrically regular when restricted to a full absorbing set S from Proposition 15.3.2.

Now suppose that the chain does not admit an accessible atom. If the chain is f-geometrically ergodic then it is straightforward that for every m-skeleton and every x we have

$$\sum_n r^n\,|P^{nm}(x,f)-\pi(f)|<\infty.$$

and for the split chain corresponding to one such skeleton we also have $|r^n\,\check{P}^n(x,f)-\pi(f)|$ summable. From the first part of the proof this ensures that the split chain, and again trivially the m-skeleton is $f^{(m)}$-geometrically regular, at least on a full absorbing set S. We can then use Theorem 15.3.7 to deduce that the original chain is f-geometrically regular on S as required. □

One of the uses of this result is to show that even when $\pi(f)<\infty$ there is no guarantee that geometric ergodicity actually implies f-geometric ergodicity: rates of convergence need not be inherited by the f-norm convergence for "large" functions f. We will see this in the example defined by (16.24) in the next chapter.

However, we can show that local geometric ergodicity does at least give the V-geometric ergodicity of Theorem 15.4.1, for an appropriate V. As in Chapter 13, we conclude with what is now an easy result.

Theorem 15.4.3 *Suppose that $\boldsymbol{\Phi}$ is an aperiodic positive Harris chain, with invariant probability measure π, and that there exists some ν-small set $C\in\mathcal{B}^+(\mathsf{X})$, $\rho_C<1$ and $M_C<\infty$, and $P^\infty(C)>0$ such that $\nu(C)>0$ and*

$$|\int_C \nu_C(dx)(P^n(x,C)-P^\infty(C))|\leq M_C\rho_C^n \tag{15.47}$$

where $\nu_C(\,\cdot\,)=\nu(\,\cdot\,)/\nu(C)$ is normalized to a probability measure on C.

Then there exists a full absorbing set S such that the chain restricted to S is geometrically ergodic.

PROOF Using the Nummelin splitting via the set C for the m-skeleton, we have exactly as in the proof of Theorem 13.3.5 that the bound (15.47) implies that the atom in the skeleton chain split at C is geometrically ergodic.

We can then emulate step (iii) of the proof of Theorem 15.4.1 above to reach the conclusion. □

Notice again that (15.47) is implied by (15.1), so that we have completed the circle of results in Theorem 15.0.1.

15.5 Simple random walk and linear models

In order to establish geometric ergodicity for specific models, we will of course use the drift criterion (V4) as a practical tool to establish the required properties of the chain.

We conclude by illustrating this for three models: the simple random walk on $\mathbb{Z}_+$, the simple linear model, and a bilinear model. We give many further examples in Chapter 16, after we have established a variety of desirable and somewhat surprising consequences of geometric ergodicity.

15.5.1 Bernoulli random walk

Consider the simple random walk on $\mathbb{Z}_+$ with transition law

$$P(x, x+1) = p,\ x \geq 0; \qquad P(x, x-1) = 1-p,\ x > 0; \qquad P(0,0) = 1-p.$$

For this chain we can consider directly $\mathsf{P}_x(\tau_0 = n) = a_x(n)$ in order to evaluate the geometric tails of the distribution of the hitting times. Since we have the recurrence relations

$$\begin{array}{rcl} a_x(n) & = & (1-p)a_{x-1}(n-1) + pa_{x+1}(n-1), \qquad x > 1; \\ a_x(0) & = & 0, \qquad x \geq 1; \\ a_1(n) & = & pa_2(n-1), \qquad a_0(0) = 0, \end{array}$$

valid for $n \geq 1$, the generating functions $A_x(z) = \sum_{n=0}^{\infty} a_x(n) z^n$ satisfy

$$\begin{array}{rcl} A_x(z) & = & z(1-p)A_{x-1}(z) + zpA_{x+1}(z), \qquad x > 1; \\ A_1(z) & = & z(1-p) + zpA_2(z), \end{array}$$

giving the solution

$$A_x(z) = \Big[\frac{1-(1-4pqz^2)^{1/2}}{2pz}\Big]^x = \big[A_1(z)\big]^x. \tag{15.48}$$

This is analytic for $z < 2/\sqrt{p(1-p)}$, so that if $p < 1/2$ (that is, if the chain is ergodic) then the chain is also geometrically ergodic.

Using the drift criterion (V4) to establish this same result is rather easier. Consider the test function $V(x) = z^x$ with $z > 1$. Then we have, for $x > 0$,

$$\Delta V(x) = z^x[(1-p)z^{-1} + pz - 1]$$

and if $p < 1/2$, then $[(1-p)z^{-1} + pz - 1] = -\beta < 0$ for z sufficiently close to unity, and so (15.28) holds as desired.

In fact, this same property, that for random walks on the half line ergodic chains are also geometrically ergodic, holds in much wider generality. The crucial property is that the increment distribution have exponentially decreasing right tails, as we shall see in Section 16.1.3.

15.5.2 Autoregressive and bilinear models

Models common in time series, especially those with some autoregressive character, often converge geometrically quickly without the need to assume that the innovation distribution has exponential character. This is because the exponential "drift" of such models comes from control of the autoregressive terms, which "swamp" the linear drift of the innovation terms for large state space values. Thus the linear or quadratic functions used to establish simple ergodicity will satisfy the Foster criterion (V2), not merely in a linear way as is the case of random walk, but in fact in the stronger mode necessary to satisfy (15.28).

We will therefore often find that, for such models, we have already established geometric ergodicity by the steps used to establish simple ergodicity or even boundedness in probability, with no further assumptions on the structure of the model.

Simple linear models Consider again the simple linear model defined in (SLM1) by

$$X_n = \alpha X_{n-1} + W_n$$

and assume W has an everywhere positive density so the chain is a ψ-irreducible T-chain. Now choosing $V(x) = |x| + 1$ gives

$$\mathsf{E}_x[V(X_1)] \leq |\alpha| V(x) + \mathsf{E}[|W|] + 1. \tag{15.49}$$

We noted in Proposition 11.4.2 that for large enough m, V satisfies (V2) with $C = C_V(m) = \{x : |x| + 1 \leq m\}$, provided that

$$\mathsf{E}[|W|] < \infty, \qquad |\alpha| < 1 :$$

thus $\{X_n\}$ admits an invariant probability measure under these conditions.

But now we can look with better educated eyes at (15.49) to see that V is in fact a solution to (15.28) under precisely these same conditions, and so we can strengthen Proposition 11.4.2 to give the conclusion that such simple linear models are geometrically ergodic.

Scalar bilinear models We illustrate this phenomenon further by re-considering the scalar bilinear model, and examining the conditions which we showed in Section 12.5.2 to be sufficient for this model to be bounded in probability. Recall that $\mathbf{X}$ is defined by the bilinear process on $\mathsf{X} = \mathbb{R}$

$$X_{k+1} = \theta X_k + b W_{k+1} X_k + W_{k+1} \tag{15.50}$$

where $\mathbf{W}$ is i.i.d. From Proposition 7.1.3 we know when $\boldsymbol{\Phi}$ is a T-chain.

To obtain a geometric rate of convergence, we reinterpret (12.36) which showed that

$$\mathsf{E}[|X_{k+1}| \mid X_k = x] \leq \mathsf{E}[|\theta + b W_{k+1}|]|x| + \mathsf{E}[|W_{k+1}|] \tag{15.51}$$

to see that $V(x) = |x| + 1$ is a solution to (V4) provided that

$$\mathsf{E}[|\theta + bW_{k+1}|] < 1. \tag{15.52}$$

Under this condition, just as in the simple linear model, the chain is irreducible and aperiodic and thus again in this case we have that the chain is V-geometrically ergodic with $V(x) = |x| + 1$.

Suppose further that $\mathbf{W}$ has finite variance σ_w^2 satisfying

$$\theta^2 + b^2\sigma_w^2 < 1;$$

exactly as in Section 14.4.2, we see that $V(x) = x^2$ is a solution to (V4) and hence $\boldsymbol{\Phi}$ is V-geometrically ergodic with this V. As a consequence, the chain admits a second order stationary distribution π with the property that for some $r > 1$ and $c < \infty$, and all x and n,

$$\sum_n r^n \Big| \int P^n(x, dy)y^2 - \int \pi(dy)y^2 \Big| < c(x^2 + 1).$$

Thus not only does the chain admit a second order stationary version, but the time dependent variances converge to the stationary variance.

15.6 Commentary

Unlike much of the ergodic theory of Markov chains, the history of geometrically ergodic chains is relatively straightforward. The concept was introduced by Kendall in [130], where the existence of the solidarity property for countable space chains was first established: that is, if one transition probability sequence $P^n(i,i)$ converges geometrically quickly, so do all such sequences. In this seminal paper the critical renewal theorem (Theorem 15.1.1) was established.

The central result, the existence of the common convergence rate, is due to Vere-Jones [281] in the countable space case; the fact that no common best bound exists was also shown by Vere-Jones [281], with the more complex example given in Section 15.1.4 being due to Teugels [263]. Vere-Jones extended much of this work to non-negative matrices [283, 284], and this approach carries over to general state space operators [272, 273, 202].

Nummelin and Tweedie [206] established the general state space version of geometric ergodicity, and by using total variation norm convergence, showed that there is independence of A in the bounds on $|P^n(x, A) - \pi(A)|$, as well as an independent geometric rate. These results were strengthened by Nummelin and Tuominen [204], who also show as one important application that it is possible to use this approach to establish geometric rates of convergence in the Key Renewal Theorem of Section 14.5 if the increment distribution has geometric tails. Their results rely on a geometric trials argument to link properties of skeletons and chains: the drift condition approach here is new, as is most of the geometric regularity theory.

The upper bound in (15.4) was first observed by Chan [42]. In Meyn and Tweedie [178], the f-geometric ergodicity approach is developed, thus leading to the final form of Theorem 15.4.1; as discussed in the next chapter, this form has important operator-theoretic consequences, as pointed out in the case of countable X by Hordijk and Spieksma [99].

The drift function criterion was first observed by Popov [218] for countable chains, with general space versions given by Nummelin and Tuominen [204] and Tweedie [278]. The full set of equivalences in Theorem 15.0.1 is new, although much of it is implicit in Nummelin and Tweedie [206] and Meyn and Tweedie [178].

Initial application of the results to queueing models can be found in Vere-Jones [282] and Miller [185], although without the benefit of the drift criteria, such applications are hard work and restricted to rather simple structures. The bilinear model in Section 15.5.2 is first analyzed in this form in Feigin and Tweedie [74]. Further interpretation and exploitation of the form of (15.4) is given in the next chapter, where we also provide a much wider variety of applications of these results.

In general, establishing exact rates of convergence or even bounds on such rates remains (for infinite state spaces) an important open problem, although by analyzing Kendall's Theorem in detail Spieksma [254] has recently identified upper bounds on the area of convergence for some specific queueing models.

Added in Second Printing There has now been a substantial amount of work on this problem, and quite different methods of bounding the convergence rates have been found by Meyn and Tweedie [183], Baxendale [17], Rosenthal [232] and Lund and Tweedie [157]. However, apart from the results in [157] which apply only to stochastically monotone chains, none of these bounds are tight, and much remains to be done in this area.