11

Drift and Regularity

Using the finiteness of the invariant measure to classify two different levels of stability is intuitively appealing. It is simple, and it also involves a fundamental stability requirement of many classes of models. Indeed, in time series analysis for example, a standard starting point, rather than an end-point, is the requirement that the model be stationary, and it follows from (10.4) that for a stationary version of a model to exist we are in effect requiring that the structure of the model be positive recurrent.

In this chapter we consider two other descriptions of positive recurrence which we show to be equivalent to that involving finiteness of π.

The first is in terms of regular sets.

> **Regularity**
>
> A set $C \in \mathcal{B}(\mathsf{X})$ is called *regular* when $\boldsymbol{\Phi}$ is ψ-irreducible, if
>
> $$\sup_{x \in C} \mathsf{E}_x[\tau_B] < \infty, \qquad B \in \mathcal{B}^+(\mathsf{X}) \tag{11.1}$$
>
> The chain $\boldsymbol{\Phi}$ is called *regular* if there is a countable cover of X by regular sets.

We know from Theorem 10.2.1 that when there is a finite invariant measure and an atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$ then $\mathsf{E}_\alpha[\tau_\alpha] < \infty$. A regular set $C \in \mathcal{B}^+(\mathsf{X})$ as defined by (11.1) has the property not only that the return times to C itself, but indeed the mean hitting times on any set in $\mathcal{B}^+(\mathsf{X})$ are bounded from starting points in C.

We will see that there is a second, equivalent, approach in terms of conditions on the one-step "mean drift"

$$\Delta V(x) = \int_{\mathsf{X}} P(x, dy) V(y) - V(x) = \mathsf{E}_x[V(\Phi_1) - V(\Phi_0)]. \tag{11.2}$$

We have already shown in Chapter 8 and Chapter 9 that for ψ-irreducible chains, drift towards a petite set implies that the chain is recurrent or Harris recurrent, and drift away from such a set implies that the chain is transient. The high points in this chapter are the following much more wide ranging equivalences.

Theorem 11.0.1 *Suppose that $\boldsymbol{\Phi}$ is a Harris recurrent chain, with invariant measure π. Then the following three conditions are equivalent:*

(i) *The measure π has finite total mass;*

(ii) *There exists some petite set $C \in \mathcal{B}(\mathsf{X})$ and $M_C < \infty$ such that*

$$\sup_{x \in C} \mathsf{E}_x[\tau_C] \leq M_C; \tag{11.3}$$

(iii) *There exists some petite set C and some extended valued, non-negative test function V, which is finite for at least one state in X, satisfying*

$$\Delta V(x) \leq -1 + b\mathbb{1}_C(x), \qquad x \in \mathsf{X}. \tag{11.4}$$

When (iii) holds then V is finite on an absorbing full set S and the chain restricted to S is regular; and any sublevel set of V satisfies (11.3).

PROOF That (ii) is equivalent to (i) is shown by combining Theorem 10.4.10 with Theorem 11.1.4, which also shows that some full absorbing set exists on which $\boldsymbol{\Phi}$ is regular. The equivalence of (ii) and (iii) is in Theorem 11.3.11, whilst the identification of the set S as the set where V is finite is in Proposition 11.3.13, where we also show that sublevel sets of V satisfy (11.3). □

Both of these approaches, as well as giving more insight into the structure of positive recurrent chains, provide tools for further analysis of asymptotic properties in Part III.

In this chapter, the equivalence of existence of solutions of the drift condition (11.4) and the existence of regular sets is motivated, and explained to a large degree, by the deterministic results in Section 11.2. Although there are a variety of proofs of such results available, we shall develop a particularly powerful approach via a discrete time form of Dynkin's Formula.

Because it involves only the one-step transition kernel, (11.4) provides an invaluable practical criterion for evaluating the positive recurrence of specific models: we illustrate this in Section 11.4.

There exists a matching, although less important, criterion for the chain to be non-positive rather than positive: we shall also prove in Section 11.5.1 that if a test function satisfies the reverse drift condition

$$\Delta V(x) \geq 0, \qquad x \in C^c, \tag{11.5}$$

then provided the increments are bounded in mean, in the sense that

$$\sup_{x \in \mathsf{X}} \int P(x, dy)|V(x) - V(y)| < \infty \tag{11.6}$$

then the mean hitting times $\mathsf{E}_x[\tau_C]$ are infinite for $x \in C^c$.

Prior to considering drift conditions, in the next section we develop through the use of the Nummelin splitting technique the structural results which show why (11.3) holds for some petite set C, and why this "local" bounded mean return time gives bounds on the mean first entrance time to any set in $\mathcal{B}^+(\mathsf{X})$.

11.1 Regular chains

On a countable space we have a simple connection between the concept of regularity and positive recurrence.

Proposition 11.1.1 *For an irreducible chain on a countable space, positive recurrence and regularity are equivalent.*

PROOF Clearly, from Theorem 10.2.2, positive recurrence is implied by regularity. To see the converse note that, for any fixed states $x, y \in \mathsf{X}$ and any n

$$\mathsf{E}_x[\tau_x] \geq {}_xP^n(x,y)[\mathsf{E}_y[\tau_x] + n].$$

Since the left hand side is finite for any x, and by irreducibility for any y there is some n with ${}_xP^n(x,y) > 0$, we must have $\mathsf{E}_y[\tau_x] < \infty$ for all y also. □

It will require more work to find the connections between positive recurrence and regularity in general.

It is not implausible that positive chains might admit regular sets. It follows immediately from (10.33) that in the positive recurrent case for any $A \in \mathcal{B}^+(\mathsf{X})$ we have

$$\mathsf{E}_x[\tau_A] < \infty, \qquad \text{a.e. } x \in A \ [\pi] \tag{11.7}$$

Thus we have from the form of π more than enough "almost-regular" sets in the positive recurrent case.

To establish the existence of true regular sets we first consider ψ-irreducible chains which possess a recurrent atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$. Although it appears that regularity may be a difficult criterion to meet since in principle it is necessary to test the hitting time of every set in $\mathcal{B}^+(\mathsf{X})$, when an atom exists it is only necessary to consider the first hitting time to the atom.

Theorem 11.1.2 *Suppose that there exists an accessible atom $\boldsymbol{\alpha} \in \mathcal{B}^+(\mathsf{X})$.*

(i) *If $\boldsymbol{\Phi}$ is positive recurrent then there exists a decomposition*

$$\mathsf{X} = S \cup N \tag{11.8}$$

where the set S is full and absorbing, and $\boldsymbol{\Phi}$ restricted to S is regular.

(ii) *The chain $\boldsymbol{\Phi}$ is regular if and only if*

$$\mathsf{E}_x[\tau_\alpha] < \infty \tag{11.9}$$

for every $x \in \mathsf{X}$.

PROOF Let

$$S := \{x : \mathsf{E}_x[\tau_\alpha] < \infty\};$$

obviously S is absorbing, and since the chain is positive recurrent we have from Theorem 10.4.10 (ii) that $\mathsf{E}_\alpha[\tau_\alpha] < \infty$, and hence $\boldsymbol{\alpha} \in S$. This also shows immediately that S is full by Proposition 4.2.3.

Let B be any set in $\mathcal{B}^+(\mathsf{X})$ with $B \subseteq \boldsymbol{\alpha}^c$, so that for π-almost all $y \in B$ we have $\mathsf{E}_y[\tau_B] < \infty$ from (11.7). From ψ-irreducibility there must then exist amongst these values one w and some n such that ${}_BP^n(w, \boldsymbol{\alpha}) > 0$. Since

$$\mathsf{E}_w[\tau_B] \geq {}_B P^n(w, \boldsymbol{\alpha}) \mathsf{E}_\alpha[\tau_B]$$

we must have $\mathsf{E}_\alpha[\tau_B] < \infty$.

Let us set

$$S_n = \{y : \mathsf{E}_y[\tau_\alpha] \leq n\}. \tag{11.10}$$

We have the obvious inequality for any x and any $B \in \mathcal{B}^+(\mathsf{X})$ that

$$\mathsf{E}_x[\tau_B] \leq \mathsf{E}_x[\tau_\alpha] + \mathsf{E}_\alpha[\tau_B] \tag{11.11}$$

so that each S_n is a regular set, and since $\{S_n\}$ is a cover of S, we have that $\boldsymbol{\Phi}$ restricted to S is regular.

This proves (i): to see (ii) note that under (11.9) we have $\mathsf{X} = S$, so the chain is regular; whilst the converse is obvious. □

It is unfortunate that the ψ-null set N in Theorem 11.1.2 need not be empty. For consider a chain on $\mathbb{Z}_+$ with

$$\begin{aligned} P(0,0) &= 1 \\ P(j,0) &= \beta_j > 0 \\ P(j,j+1) &= 1-\beta_j. \end{aligned} \tag{11.12}$$

Then the chain restricted to $\{0\}$ is trivially regular, and the whole chain is positive recurrent; but if

$$\sum_j \prod_1^j \beta_k = \infty$$

then the chain is not regular, and $N = \{1, 2, \ldots\}$ in (11.8).

It is the weak form of irreducibility we use which allows such null sets to exist: this pathology is of course avoided on a countable space under the normal form of irreducibility, as we saw in Proposition 11.1.1.

However, even under ψ-irreducibility we can extend this result without requiring an atom in the original space.

Let us next consider the case where $\boldsymbol{\Phi}$ is strongly aperiodic, and use the Nummelin splitting to define $\check{\boldsymbol{\Phi}}$ on $\check{\mathsf{X}}$ as in Section 5.1.1.

Proposition 11.1.3 *Suppose that $\boldsymbol{\Phi}$ is strongly aperiodic and positive recurrent. Then there exists a decomposition*

$$\mathsf{X} = S \cup N \tag{11.13}$$

where the set S is full and absorbing, and $\boldsymbol{\Phi}$ restricted to S is regular.

Proof We know from Proposition 10.4.2 that the split chain is also positive recurrent with invariant probability measure $\check{\pi}$; and thus for $\check{\pi}$-a.e. $x_i \in \check{\mathsf{X}}$, by (11.7) we have that

$$\check{\mathsf{E}}_{x_i}[\tau_{\check{\alpha}}] < \infty. \tag{11.14}$$

Let $\check{S} \subseteq \check{\mathsf{X}}$ denote the set where (11.14) holds. Then it is obvious that $\check{S}$ is absorbing, and by Theorem 11.1.2 the chain $\check{\boldsymbol{\Phi}}$ is regular on $\check{S}$. Let $\{\check{S}_n\}$ denote the cover of $\check{S}$ with regular sets.

Now we have $\check{N} = \check{\mathsf{X}} \backslash \check{S} \subseteq \mathsf{X}_0$, and so if we write N as the copy of $\check{N}$ and define $S = \mathsf{X} \backslash N$, we can cover S with the matching copies S_n. We then have for $x \in S_n$ and any $B \in \mathcal{B}^+(\mathsf{X})$

$$\mathsf{E}_x[\tau_B] \leq \check{\mathsf{E}}_{x_0}[\tau_B] + \check{\mathsf{E}}_{x_1}[\tau_B]$$

which is bounded for $x_0 \in \check{S}_n$ and all $x_1 \in \check{\boldsymbol{\alpha}}$, and hence for $x \in S_n$.

Thus S is the required full absorbing set for (11.13) to hold. □

It is now possible, by the device we have used before of analyzing the m-skeleton, to show that this proposition holds for arbitrary positive recurrent chains.

Theorem 11.1.4 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible. Then the following are equivalent:*

(i) *The chain $\boldsymbol{\Phi}$ is positive recurrent.*

(ii) *There exists a decomposition*

$$\mathsf{X} = S \cup N \tag{11.15}$$

where the set S is full and absorbing, and $\boldsymbol{\Phi}$ restricted to S is regular.

PROOF Assume $\boldsymbol{\Phi}$ is positive recurrent. Then the Nummelin splitting exists for some m-skeleton from Proposition 5.4.5, and so we have from Proposition 11.1.3 that there is a decomposition as in (11.15) where the set $S = \cup S_n$ and each S_n is regular for the m-skeleton.

But if τ_B^m denotes the number of steps needed for the m-skeleton to reach B, then we have that

$$\tau_B \leq m\ \tau_B^m$$

and so each S_n is also regular for $\boldsymbol{\Phi}$ as required.

The converse is almost trivial: when the chain is regular on S then there exists a petite set C inside S with $\sup_{x \in C} \mathsf{E}_x[\tau_C] < \infty$, and the result follows from Theorem 10.4.10. □

Just as we may restrict any recurrent chain to an absorbing set H on which the chain is Harris recurrent, we have here shown that we can further restrict a positive recurrent chain to an absorbing set where it is regular.

We will now turn to the equivalence between regularity and mean drift conditions. This has the considerable benefit that it enables us to identify exactly the null set on which regularity fails, and thus to eliminate from consideration annoying and pathological behavior in many models. It also provides, as noted earlier, a sound practical approach to assessing stability of the chain.

To motivate and perhaps give more insight into the connections between hitting times and mean drift conditions we first consider deterministic models.

11.2 Drift, hitting times and deterministic models

In this section we analyze a deterministic state space model, indicating the role we might expect the drift conditions (11.4) on ΔV to play. As we have seen in Chapter 4 and Chapter 7 in examining irreducibility structures, the underlying deterministic models for state space systems foreshadow the directions to be followed for systems with a noise component.

Let us then assume that there is a topology on X, and consider the deterministic process known as a semi-dynamical system.

The Semi-Dynamical System

(DS1) The process $\boldsymbol{\Phi}$ is deterministic, and generated by the nonlinear difference equation, or semi-dynamical system,

$$\Phi_{k+1} = F(\Phi_k), \qquad k \in \mathbb{Z}_+, \tag{11.16}$$

where $F\colon \mathsf{X} \to \mathsf{X}$ is a continuous function.

Although $\boldsymbol{\Phi}$ is deterministic, it is certainly a Markov chain (if a trivial one in a probabilistic sense), with Markov transition operator P defined through its operations on any function f on X by

$$Pf(\,\cdot\,) = f(F(\,\cdot\,)).$$

Since we have assumed the function F to be continuous, the Markov chain $\boldsymbol{\Phi}$ has the Feller property, although in general it will not be a T-chain.

For such a deterministic system it is standard to consider two forms of stability known as recurrence and ultimate boundedness. We shall call the deterministic system (11.16) *recurrent* if there exists a compact subset $C \subset \mathsf{X}$ such that $\sigma_C(x) < \infty$ for each initial condition $x \in \mathsf{X}$. Such a concept of recurrence here is almost identical to the definition of recurrence for stochastic models. We shall call the system (11.16) *ultimately bounded* if there exists a compact set $C \subset \mathsf{X}$ such that for each fixed initial condition $\Phi_0 \in \mathsf{X}$, the trajectory starting at Φ_0 eventually enters and remains in C. Ultimate boundedness is loosely related to positive recurrence: it requires that the limit points of the process all lie within a compact set C, which is somewhat analogous to the positivity requirement that there be an invariant probability measure π with $\pi(C) > 1 - \varepsilon$ for some small ε.

Drift Condition for the Semi-dynamical System

(DS2) There exists a positive function $V\colon \mathsf{X} \to \mathbb{R}_+$ and a compact set $C \subset \mathsf{X}$ and constant $M < \infty$ such that

$$\Delta V(x) := V(F(x)) - V(x) \le -1$$

for all x lying outside the compact set C, and

$$\sup_{x \in C} V(F(x)) \le M.$$

If we consider the sequence $V(\Phi_n)$ on $\mathbb{R}_+$ then this condition requires that this sequence move monotonically downwards at a uniform rate until the first time that $\boldsymbol{\Phi}$ enter C. It is therefore not surprising that $\boldsymbol{\Phi}$ hits C in a finite time under this condition.

Theorem 11.2.1 *Suppose that* $\boldsymbol{\Phi}$ *is defined by (DS1).*

(i) *If (DS2) is satisfied, then* $\boldsymbol{\Phi}$ *is ultimately bounded.*

(ii) *If* $\boldsymbol{\Phi}$ *is recurrent, then there exists a positive function* V *such that (DS2) holds.*

(iii) *Hence* $\boldsymbol{\Phi}$ *is recurrent if and only if it is ultimately bounded.*

PROOF To prove (i), let $\Phi(x,n) = F^n(x)$ denote the deterministic position of Φ_n if the chain starts at $\Phi_0 = x$. We first show that the compact set C' defined as

$$C' := \bigcup \{\Phi(x,i) : x \in C,\ 1 \le i \le M+1\} \cup C$$

where M is the constant used in (DS2), is invariant as defined in Chapter 7.

For any $x \in C$ we have $\Phi(x,i) \in C$ for some $1 \le i \le M+1$ by (DS2) and the hypothesis that V is positive. Hence for an arbitrary $j \in \mathbb{Z}_+$, $\Phi(x,j) = \Phi(y,i)$ for some $y \in C$, and some $1 \le i \le M+1$. This implies that $\Phi(x,j) \in C'$ and hence C' is equal to the invariant set

$$C' = \bigcup_{i=1}^{\infty} \{\Phi(x,i) : x \in C\} \cup C.$$

Because V is positive and decreases on C^c, every trajectory must enter the set C, and hence also C' at some finite time. We conclude that $\boldsymbol{\Phi}$ is ultimately bounded.

We now prove (ii). Suppose that a compact set C_1 exists such that $\sigma_{C_1}(x) < \infty$ for each initial condition $x \in \mathsf{X}$. Let O be an open pre-compact set containing C_1, and set $C := cl\, O$. Then the test function

$$V(x) := \sigma_O(x)$$

satisfies (DS2). To see this, observe that if $x \in C^c$, then $V(F(x)) = V(x) - 1$ and hence the first inequality is satisfied. By assumption the function V is everywhere finite, and since O is open it follows that V is upper semicontinuous from Proposition 6.1.1. This implies that the second inequality in (DS2) holds, since a finite-valued upper semicontinuous function is uniformly bounded on compact sets. □

For a semi-dynamical system, this result shows that recurrence is actually equivalent to ultimate boundedness. In this the deterministic system differs from the general NSS(F) model with a non-trivial random component. More pertinently, we have also shown that the semi-dynamical system is ultimately bounded if and only if a test function exists satisfying (DS2).

This test function may always be taken to be the time to reach a certain compact set. As an almost exact analogue, we now go on to see that the expected time to reach a petite set is the appropriate test function to establish positive recurrence in the stochastic framework; and that, as we show in Theorem 11.3.4 and Theorem 11.3.5, the existence of a test function similar to (DS2) is equivalent to positive recurrence.

11.3 Drift criteria for regularity

11.3.1 Mean drift and Dynkin's Formula

The deterministic models of the previous section lead us to hope that we can obtain criteria for regularity by considering a drift criterion for positive recurrence based on (11.4).

What is somewhat more surprising is the depth of these connections and the direct method of attack on regularity which we have through this route.

The key to exploiting the effect of mean drift is the following condition, which is stronger on C^c than (V1) and also requires a bound on the drift away from C.

Strict Drift Towards C

(V2) For some set $C \in \mathcal{B}(\mathsf{X})$, some constant $b < \infty$, and an extended real-valued function $V: \mathsf{X} \to [0, \infty]$

$$\Delta V(x) \leq -1 + b\mathbb{1}_C(x) \qquad x \in \mathsf{X}. \tag{11.17}$$

This is a portmanteau form of the following two equations:

$$\Delta V(x) \leq -1, \qquad x \in C^c, \tag{11.18}$$

for some non-negative function V and some set $C \in \mathcal{B}(\mathsf{X})$; and for some $M < \infty$,

$$\Delta V(x) \leq M, \qquad x \in C. \tag{11.19}$$

Thus we might hope that (V2) might have something of the same impact for stochastic models as (DS2) has for deterministic chains.

In essentially the form (11.18) and (11.19) these conditions were introduced by Foster [82] for countable state space chains, and shown to imply positive recurrence. Use of the form (V2) will actually make it easier to show that the existence of everywhere finite solutions to (11.17) is equivalent to regularity and moreover we will identify the sublevel sets of the test function V as regular sets.

The central technique we will use to make connections between one-step mean drifts and moments of first entrance times to appropriate (usually petite) sets hinges on a discrete time version of a result known for continuous time processes as Dynkin's Formula.

This formula yields not only those criteria for positive Harris chains and regularity which we discuss in this chapter, but also leads in due course to necessary and sufficient conditions for rates of convergence of the distributions of the process; necessary and sufficient conditions for finiteness of moments; and sample path ergodic theorems such as the Central Limit Theorem and Law of the Iterated Logarithm. All of these are considered in Part III.

Dynkin's Formula is a sample path formula, rather than a formula involving probabilistic operators. We need to introduce a little more notation to handle such situations.

Recall from Section 3.4 the definition

$$\mathcal{F}_k^{\Phi} = \sigma\{\Phi_0, \dots, \Phi_k\}, \tag{11.20}$$

and let $\{Z_k, \mathcal{F}_k^{\Phi}\}$ be an adapted sequence of positive random variables. For each k, Z_k will denote a fixed Borel measurable function of $(\Phi_0, \dots, \Phi_k)$, although in applications this will usually (although not always) be a function of the last position, so that

$$Z_k(\Phi_0, \dots, \Phi_k) = Z(\Phi_k)$$

for some measurable function Z. We will somewhat abuse notation and let Z_k denote both the random variable, and the function on X^{k+1}.

For any stopping time τ define

$$\tau^n := \min\{n, \tau, \inf\{k \ge 0 : Z_k \ge n\}\}.$$

The random time τ^n is also a stopping time since it is the minimum of stopping times, and the random variable $\sum_{i=0}^{\tau^n - 1} Z_i$ is essentially bounded by n^2.

Dynkin's Formula will now tell us that we can evaluate the expected value of Z_{τ^n} by taking the initial value Z_0 and adding on to this the average increments at each time until τ^n. This is almost obvious, but has wide-spread consequences: in particular it enables us to use (V2) to control these one-step average increments, leading to control of the expected overall hitting time.

Theorem 11.3.1 (Dynkin's Formula) *For each $x \in \mathsf{X}$ and $n \in \mathbb{Z}_+$,*

$$\mathsf{E}_x[Z_{\tau^n}] = \mathsf{E}_x[Z_0] + \mathsf{E}_x\Big[\sum_{i=1}^{\tau^n} (\mathsf{E}[Z_i \mid \mathcal{F}_{i-1}^{\Phi}] - Z_{i-1})\Big]$$

PROOF For each $n \in \mathbb{Z}_+$,

$$\begin{aligned} Z_{\tau^n} &= Z_0 + \sum_{i=1}^{\tau^n} (Z_i - Z_{i-1}) \\ &= Z_0 + \sum_{i=1}^{n} \mathbb{1}\{\tau^n \ge i\}(Z_i - Z_{i-1}) \end{aligned}$$

Taking expectations and noting that $\{\tau^n \ge i\} \in \mathcal{F}_{i-1}^{\Phi}$ we obtain

$$\begin{aligned} \mathsf{E}_x[Z_{\tau^n}] &= \mathsf{E}_x[Z_0] + \mathsf{E}_x\Big[\sum_{i=1}^{n} \mathsf{E}_x[Z_i - Z_{i-1} \mid \mathcal{F}_{i-1}^{\Phi}]\mathbb{1}\{\tau^n \ge i\}\Big] \\ &= \mathsf{E}_x[Z_0] + \mathsf{E}_x\Big[\sum_{i=1}^{\tau^n} (\mathsf{E}_x[Z_i \mid \mathcal{F}_{i-1}^{\Phi}] - Z_{i-1})\Big] \end{aligned}$$

□

As an immediate corollary we have

Proposition 11.3.2 *Suppose that there exist two sequences of positive functions* $\{s_k, f_k : k \geq 0\}$ *on* X*, such that*

$$\mathsf{E}[Z_{k+1} \mid \mathcal{F}_k^{\Phi}] \leq Z_k - f_k(\Phi_k) + s_k(\Phi_k).$$

Then for any initial condition x *and any stopping time* τ

$$\mathsf{E}_x[\sum_{k=0}^{\tau-1} f_k(\Phi_k)] \leq Z_0(x) + \mathsf{E}_x[\sum_{k=0}^{\tau-1} s_k(\Phi_k)].$$

Proof Fix $N > 0$ and note that

$$\mathsf{E}[Z_{k+1} \mid \mathcal{F}_k^{\Phi}] \leq Z_k - f_k(\Phi_k) \wedge N + s_k(\Phi_k).$$

By Dynkin's Formula

$$0 \leq \mathsf{E}_x[Z_{\tau^n}] \leq Z_0(x) + \mathsf{E}_x\Big[\sum_{i=1}^{\tau^n}(s_{i-1}(\Phi_{i-1}) - [f_{i-1}(\Phi_{i-1}) \wedge N])\Big]$$

and hence by adding the finite term

$$\mathsf{E}_x\Big[\sum_{k=1}^{\tau^n}[f_{k-1}(\Phi_{k-1}) \wedge N]\Big]$$

to each side we get

$$\mathsf{E}_x\Big[\sum_{k=1}^{\tau^n}[f_{k-1}(\Phi_{k-1}) \wedge N]\Big] \leq Z_0(x) + \mathsf{E}_x\Big[\sum_{k=1}^{\tau^n} s_{k-1}(\Phi_{k-1})\Big] \leq Z_0(x) + \mathsf{E}_x\Big[\sum_{k=1}^{\tau} s_{k-1}(\Phi_{k-1})\Big].$$

Letting $n \to \infty$ and then $N \to \infty$ gives the result by the Monotone Convergence Theorem. □

Closely related to this we have

Proposition 11.3.3 *Suppose that there exists a sequence of positive functions* $\{\varepsilon_k : k \geq 0\}$ *on* X*,* $c < \infty$*, such that*

(i) $\varepsilon_{k+1}(x) \leq c\varepsilon_k(x), \qquad k \in \mathbb{Z}_+,\ x \in A^c$;

(ii) $\mathsf{E}[Z_{k+1} \mid \mathcal{F}_k^{\Phi}] \leq Z_k - \varepsilon_k(\Phi_k), \qquad \sigma_A > k.$

Then

$$\mathsf{E}_x[\sum_{i=0}^{\tau_A-1} \varepsilon_i(\Phi_i)] \leq \begin{cases} Z_0(x), & x \in A^c; \\ \varepsilon_0(x) + cPZ_0\,(x), & x \in \mathsf{X}. \end{cases}$$

Proof Let Z_k and ε_k denote the random variables $Z_k(\Phi_0, \dots, \Phi_k)$ and $\varepsilon_k(\Phi_k)$ respectively.

By hypothesis $\mathsf{E}[Z_k \mid \mathcal{F}_{k-1}^{\Phi}] - Z_{k-1} \leq -\varepsilon_{k-1}$ whenever $1 \leq k \leq \sigma_A$. Hence for all $n \in \mathbb{Z}_+$ and $x \in \mathsf{X}$ we have by Dynkin's Formula

$$0 \leq \mathsf{E}_x[Z_{\tau_A^n}] \leq Z_0(x) - \mathsf{E}_x\Big[\sum_{i=1}^{\tau_A^n} \varepsilon_{i-1}(\Phi_{i-1})\Big], \qquad x \in A^c.$$

By the Monotone Convergence Theorem it follows that for all initial conditions,

$$\mathsf{E}_x\Big[\sum_{i=1}^{\tau_A} \varepsilon_{i-1}(\Phi_{i-1})\Big] \le Z_0(x) \qquad x \in A^c.$$

This proves the result for $x \in A^c$.

For arbitrary x we have

$$\begin{aligned}\mathsf{E}_x\Big[\sum_{i=1}^{\tau_A} \varepsilon_{i-1}(\Phi_{i-1})\Big] &= \varepsilon_0(x) + \mathsf{E}_x\Big[\mathsf{E}_{\Phi_1}\Big(\sum_{i=1}^{\tau_A} \varepsilon_i(\Phi_{i-1})\Big)\mathbb{1}(\Phi_1 \in A^c)\Big] \\ &\le \varepsilon_0(x) + cPZ_0\,(x).\end{aligned}$$

□

We can immediately use Dynkin's Formula to prove

Theorem 11.3.4 *Suppose $C \in \mathcal{B}(\mathsf{X})$, and V satisfies (V2). Then*

$$\mathsf{E}_x[\tau_C] \le V(x) + b\mathbb{1}_C(x)$$

for all x. Hence if C is petite and V is everywhere finite and bounded on C then $\boldsymbol{\Phi}$ is positive Harris recurrent.

Proof Applying Proposition 11.3.3 with $Z_k = V(\Phi_k)$, $\varepsilon_k = 1$ we have the bound

$$\mathsf{E}_x[\tau_C] \le \begin{cases} V(x) & \text{for } x \in C^c \\ 1 + PV\,(x) & x \in C \end{cases}$$

Since (V2) gives $PV \le V - 1 + b$ on C, we have the required result.

If V is everywhere finite then this bound trivially implies $L(x, C) \equiv 1$ and so, if C is petite, the chain is Harris recurrent from Proposition 9.1.7. Positivity follows from Theorem 10.4.10 (ii). □

We will strengthen Theorem 11.3.4 below in Theorem 11.3.11 where we show that V need not be bounded on C, and moreover that (V2) gives bounds on the mean return time to general sets in $\mathcal{B}^+(\mathsf{X})$.

11.3.2 Hitting times and test functions

The upper bound in Theorem 11.3.4 is a typical consequence of the drift condition. The key observation in showing the actual equivalence of mean drift towards petite sets and regularity is the identification of specific solutions to (V2) when the chain is regular.

For any set $A \in \mathcal{B}(\mathsf{X})$ we define the kernel G_A on $(\mathsf{X}, \mathcal{B}(\mathsf{X}))$ through

$$G_A(x, f) := [I + I_{A^c}U_A]\,(x, f) = \mathsf{E}_x[\sum_{k=0}^{\sigma_A} f(\Phi_k)] \tag{11.21}$$

where x is an arbitrary state, and f is any positive function.

For $f \ge 1$ fixed we will see in Theorem 11.3.5 that the function $V = G_C(\,\cdot\,, f)$ satisfies (V2), and also a generalization of this drift condition to be developed in later chapters. In this chapter we concentrate on the special case where $f \equiv 1$ and we will simplify the notation by setting

$$V_C(x) = G_C(x, \mathsf{X}) = 1 + \mathsf{E}_x[\sigma_C]. \tag{11.22}$$

Theorem 11.3.5 *For any set $A \in \mathcal{B}(\mathsf{X})$ we have*

(i) *The kernel G_A satisfies the identity*

$$PG_A = G_A - I + I_A U_A$$

(ii) *The function $V_A(\,\cdot\,) = G_A(\,\cdot\,, \mathsf{X})$ satisfies the identity*

$$PV_A(x) = V_A(x) - 1, \qquad x \in A^c. \tag{11.23}$$

$$PV_A(x) = \mathsf{E}_x[\tau_A] - 1, \qquad x \in A. \tag{11.24}$$

Thus if $C \in \mathcal{B}^+(\mathsf{X})$ is regular, V_C is a solution to (11.17).

(iii) *The function $V = V_A - 1$ is the pointwise minimal solution on A^c to the inequalities*

$$PV(x) \le V(x) - 1, \qquad x \in A^c. \tag{11.25}$$

Proof From the definition

$$U_A := \sum_{k=0}^{\infty} (PI_{A^c})^k P$$

we see that $U_A = P + PI_{A_c}U_A = PG_A$. Since $U_A = G_A - I + I_A U_A$ we have (i), and then (ii) follows.

We have that V_A solves (11.25) from (ii); but if V is any other solution then it is pointwise larger than V_A exactly as in Theorem 11.3.4. □

We shall use repeatedly the following lemmas, which guarantee finiteness of solutions to (11.17), and which also give a better description of the structure of the most interesting solution, namely V_C.

Lemma 11.3.6 *Any solution of (11.17) is finite ψ-almost everywhere or infinite everywhere.*

Proof If V satisfies (11.17) then

$$PV\,(x) \le V(x) + b$$

for all $x \in \mathsf{X}$, and it then follows that the set $\{x : V(x) < \infty\}$ is absorbing. If this set is non-empty then it is full by Proposition 4.2.3. □

Lemma 11.3.7 *If the set C is petite, then the function $V_C(x)$ is unbounded off petite sets.*

PROOF We have from Chebyshev's inequality that for each of the sublevel sets $C_V(\ell) := \{x : V_C(x) \le \ell\}$,

$$\sup_{x \in C_V(\ell)} \mathsf{P}_x\{\sigma_C \ge n\} \le \frac{\ell}{n}.$$

Since the right hand side is less than $\frac{1}{2}$ for sufficiently large n, this shows that $C_V(\ell) \overset{a}{\rightsquigarrow} C$ for a sampling distribution a, and hence, by Proposition 5.5.4, the set $C_V(\ell)$ is petite. □

Lemma 11.3.7 will typically be applied to show that a given petite set is regular. The converse is always true, as the next result shows:

Proposition 11.3.8 *If the set A is regular then it is petite.*

PROOF Again we apply Chebyshev's inequality. If $C \in \mathcal{B}^+(\mathsf{X})$ is petite then

$$\sup_{x \in A} \mathsf{P}_x\{\sigma_C > n\} \le \frac{1}{n} \sup_{x \in A} \mathsf{E}_x[\tau_C]$$

As in the proof of Lemma 11.3.7 this shows that A is petite if it is regular. □

11.3.3 Regularity, drifts and petite sets

In this section, using the full force of Dynkin's Formula and the form (V2) for the drift condition, we will find we can do rather more than bound the return times to C from states in C. We have first

Lemma 11.3.9 *If (V2) holds then for each $x \in \mathsf{X}$ and any set $B \in \mathcal{B}(\mathsf{X})$*

$$\mathsf{E}_x[\tau_B] \le V(x) + b\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B - 1} \mathbb{1}_C(\Phi_k)\Big]. \tag{11.26}$$

PROOF This follows from Proposition 11.3.2 on letting $f_k = 1$, $s_k = b\mathbb{1}_C$. □

Note that Theorem 11.3.4 is the special case of this result when $B = C$.

In order to derive the central characterization of regularity, we first need an identity linking sampling distributions and hitting times on sets.

Lemma 11.3.10 *For any first entrance time τ_B, any sampling distribution a, and any positive function $f : \mathsf{X} \to \mathbb{R}_+$, we have*

$$\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B - 1} K_a(\Phi_k, f)\Big] = \sum_{i=0}^{\infty} a_i \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B - 1} f(\Phi_{k+i})\Big].$$

PROOF By the Markov property and Fubini's Theorem we have

$$\begin{aligned}
&\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B - 1} K_a(\Phi_k, f)\Big] \\
&= \sum_{i=0}^{\infty} a_i \mathsf{E}_x\Big[\sum_{k=0}^{\infty} P^i(\Phi_k, f)\mathbb{1}\{k < \tau_B\}\Big] \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_i \mathsf{E}_x\Big[\mathsf{E}\Big[f(\Phi_{k+i}) \mid \mathcal{F}_k\Big]\mathbb{1}\{k < \tau_B\}\Big]
\end{aligned}$$

But now we have that $\mathbb{1}(k < \tau_B)$ is measurable with respect to $\mathcal{F}_k$ and so by the smoothing property of expectations this becomes

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_i \mathsf{E}_x\Big[\mathsf{E}\Big[f(\Phi_{k+i})\mathbb{1}\{k<\tau_B\} \mid \mathcal{F}_k\Big]\Big] \\
&\quad = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_i \mathsf{E}_x\Big[f(\Phi_{k+i})\mathbb{1}(k<\tau_B)\Big] \\
&\quad = \sum_{i=0}^{\infty} a_i \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} f(\Phi_{k+i})\Big].
\end{aligned}$$

□

We now have a relatively simple task in proving

Theorem 11.3.11 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible.*

(i) *If (V2) holds for a function V and a petite set C then for any $B \in \mathcal{B}^+(\mathsf{X})$ there exists $c(B) < \infty$ such that*

$$\mathsf{E}_x[\tau_B] \le V(x) + c(B), \qquad x \in \mathsf{X}.$$

Hence if V is bounded on A, then A is regular.

(ii) *If there exists one regular set $C \in \mathcal{B}^+(\mathsf{X})$, then C is petite and the function $V = V_C$ satisfies (V2), with V uniformly bounded on A for any regular set A.*

PROOF To prove (i), suppose that (V2) holds, with V bounded on A and C a ψ_a-petite set. Without loss of generality, from Proposition 5.5.6 we can assume $\sum_{i=0}^{\infty} i\, a_i < \infty$. We also use the simple but critical bound from the definition of petiteness:

$$\mathbb{1}_C(x) \le \psi_a(B)^{-1} K_a(x,B), \qquad x \in \mathsf{X}, B \in \mathcal{B}^+(\mathsf{X}). \tag{11.27}$$

By Lemma 11.3.9 and the bound (11.27) we then have

$$\begin{aligned}
\mathsf{E}_x[\tau_B] &\le V(x) + b\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} \mathbb{1}_C(\Phi_k)\Big] \\
&\le V(x) + b\mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} \psi_a(B)^{-1}K_a(\Phi_k,B)\Big] \\
&= V(x) + b\psi_a(B)^{-1}\sum_{i=0}^{\infty} a_i \mathsf{E}_x\Big[\sum_{k=0}^{\tau_B-1} \mathbb{1}_B(\Phi_{k+i})\Big] \\
&\le V(x) + b\psi_a(B)^{-1}\sum_{i=0}^{\infty}(i+1)a_i
\end{aligned}$$

for any $B \in \mathcal{B}^+(\mathsf{X})$, and all $x \in \mathsf{X}$. If V is bounded on A, it follows that

$$\sup_{x\in A} \mathsf{E}_x[\tau_B] < \infty,$$

which shows that A is regular.

To prove (ii), suppose that a regular set $C \in \mathcal{B}^+(\mathsf{X})$ exists. By Lemma 11.3.8 the set C is petite. Then $V = V_C$ is clearly positive, and bounded on any regular set A. Moreover, by Theorem 11.3.5 and regularity of C it follows that condition (V2) holds for a suitably large constant b. □

Boundedness of hitting times from arbitrary initial measures will become important in Part III. The following definition is an obvious one.

> Regularity of Measures
>
> A probability measure μ is called *regular*, if
>
> $$\mathsf{E}_\mu[\tau_B] < \infty, \qquad B \in \mathcal{B}^+(\mathsf{X})$$

The proof of the following result for regular measures μ is identical to that of the previous theorem and we omit it.

Theorem 11.3.12 *Suppose that $\boldsymbol{\Phi}$ is ψ-irreducible.*

(i) *If (V2) holds for a petite set C and a function V, and if $\mu(V) < \infty$, then the measure μ is regular.*

(ii) *If μ is regular, and if there exists one regular set $C \in \mathcal{B}^+(\mathsf{X})$, then there exists an extended-valued function V satisfying (V2) with $\mu(V) < \infty$.*

□

As an application of Theorem 11.3.11 we obtain a description of regular sets as in Theorem 11.1.4.

Proposition 11.3.13 *If there exists a regular set $C \in \mathcal{B}^+(\mathsf{X})$, then the sets $C_V(\ell) := \{x : V_C(x) \le \ell, : \ell \in \mathbb{Z}_+\}$ are regular and $S_C = \{y : V_C(y) < \infty\}$ is a full absorbing set such that $\boldsymbol{\Phi}$ restricted to S_C is regular.*

Proof Suppose that a regular set $C \in \mathcal{B}^+(\mathsf{X})$ exists. Since C is regular it is also ψ_a-petite, and we can assume without loss of generality that the sampling distribution a has a finite mean. By regularity of C we also have, by Theorem 11.3.11 (ii), that (V2) holds with $V = V_C$. From Theorem 11.3.11 each of the sets $C_V(\ell)$ is regular, and by Lemma 11.3.6 the set $S_C = \{y : V_C(y) < \infty\}$ is full and absorbing. □

Theorem 11.3.11 gives a characterization of regular sets in terms of a drift condition. Theorem 11.3.14 now gives such a characterization in terms of the mean hitting times to petite sets.

Theorem 11.3.14 *If $\boldsymbol{\Phi}$ is ψ-irreducible, then the following are equivalent:*

(i) *The set $C \in \mathcal{B}(\mathsf{X})$ is petite and $\sup_{x \in C} \mathsf{E}_x[\tau_C] < \infty$;*

(ii) *The set C is regular and $C \in \mathcal{B}^+(\mathsf{X})$.*

PROOF **(i)** Suppose that C is petite, and let as before $V_C(x) = 1 + \mathsf{E}_x[\sigma_C]$. By Theorem 11.3.5 and the conditions of the theorem we may find a constant $b < \infty$ such that

$$PV_C \le V_C - 1 + b\mathbb{1}_C.$$

Since V_C is bounded on C by construction, it follows from Theorem 11.3.11 that C is regular. Since the set C is Harris recurrent it follows from Proposition 8.3.1 (ii) that $C \in \mathcal{B}^+(\mathsf{X})$.

(ii) Suppose that C is regular. Since $C \in \mathcal{B}^+(\mathsf{X})$, it follows from regularity that $\sup_{x\in C} \mathsf{E}_x[\tau_C] < \infty$, and that C is petite follows from Proposition 11.3.8. □

We can now give the following complete characterization of the case $\mathsf{X} = S$.

Theorem 11.3.15 *Suppose that Φ is ψ-irreducible. Then the following are equivalent:*

(i) *The chain Φ is regular*

(ii) *The drift condition (V2) holds for a petite set C and an everywhere finite function V.*

(iii) *There exists a petite set C such that the expectation*

$$\mathsf{E}_x[\tau_C]$$

is finite for each x, and uniformly bounded for $x \in C$.

PROOF If (i) holds, then it follows that a regular set $C \in \mathcal{B}^+(\mathsf{X})$ exists. The function $V = V_C$ is everywhere finite and satisfies (V2), by (11.24), for a suitably large constant b; so (ii) holds. Conversely, Theorem 11.3.11 (i) tells us that if (V2) holds for a petite set C with V finite valued then each sublevel set of V is regular, and so (i) holds.

If the expectation is finite as described in (iii), then by (11.24) we see that the function $V = V_C$ satisfies (V2) for a suitably large constant b. Hence from Theorem 11.3.15 we see that the chain is regular; and the converse is trivial. □

11.4 Using the regularity criteria

11.4.1 Some straightforward applications

Random walk on a half line We have already used a drift criterion for positive recurrence, without identifying it as such, in some of our analysis of the random walk on a half line.

Using the criteria above, we have

Proposition 11.4.1 *If Φ is a random walk on a half line with finite mean increment β then Φ is regular if*

$$\beta = \int w\, \Gamma(dw) < 0;$$

and in this case all compact sets are regular sets.

PROOF By consideration of the proof of Proposition 8.5.1, we see that this result has already been established, since (11.18) was exactly the condition verified for recurrence in that case, whilst (11.19) is simply checked for the random walk. □

From the results in Section 8.5, we know that the random walk on $\mathbb{R}_+$ is transient if $\beta > 0$, and that (at least under a second moment condition) it is recurrent in the marginal case $\beta = 0$. We shall show in Proposition 11.5.3 that it is not regular in this marginal case.

11.4.1.1 Forward recurrence times We could also use this approach in a simple way to analyze positivity for the forward recurrence time chain.

In this example, using the function $V(x) = x$ we have

$$\sum_y P(x,y)V(y) = V(x) - 1, \qquad x \geq 1 \tag{11.28}$$

$$\sum_y P(0,y)V(y) = \sum_y p(y)\, y. \tag{11.29}$$

Hence, as we already know, the chain is positive recurrent if $\sum_y p(y)\, y < \infty$.

Since $\mathsf{E}_0[\tau_0] = \sum_y p(y)\, y$ the drift condition with $V(x) = x$ is also necessary, as we have seen.

The forward recurrence time chain thus provides a simple but clear example of the need to include the second bound (11.19) in the criterion for positive recurrence.

11.4.1.2 Linear models Consider the simple linear model defined in (SLM1) by

$$X_n = \alpha X_{n-1} + W_n$$

We have

Proposition 11.4.2 *Suppose that the disturbance variable W for the simple linear model defined in (SLM1), (SLM2) is non-singular with respect to Lebesgue measure, and satisfies $\mathsf{E}[\log(1 + |W|)] < \infty$. Suppose also that $|\alpha| < 1$. Then every compact set is regular, and hence the chain itself is regular.*

PROOF From Proposition 6.3.5 we know that the chain $\mathbf{X}$ is a ψ-irreducible and aperiodic T-chain under the given assumptions.

Let $V(x) = \log(1 + \varepsilon|x|)$, where $\varepsilon > 0$ will be fixed below. We will verify that (V2) holds with this choice of V by applying the following two special properties of this test function:

$$V(x+y) \leq V(x) + V(y) \tag{11.30}$$

$$\lim_{x\to\infty}[V(x) - V(|\alpha|x)] = \log((|\alpha|^{-1}) \tag{11.31}$$

From (11.30) and (SLM1),

$$V(X_1) = V(\alpha X_0 + W_1) \leq V(|\alpha| X_0) + V(W_1),$$

and hence from (11.31) there exists $r < \infty$ such that whenever $X_0 \geq r$,

$$V(X_1) \leq V(X_0) - \tfrac{1}{2}\log(|\alpha|^{-1}) + V(W_1).$$

Choosing $\varepsilon > 0$ sufficiently small so that $\mathsf{E}[V(W)] \le \frac{1}{4}\log(|\alpha|^{-1})$ we see that for $x \ge r$,

$$\mathsf{E}_x[V(X_1)] \le V(x) - \tfrac{1}{4}\log(|\alpha|^{-1}).$$

So we have that (V2) holds with $C = \{x : |x| \le r\}$ and the result follows. □

This is part of the recurrence result we proved using a stochastic comparison argument in Section 9.5.1, but in this case the direct proof enables us to avoid any restriction on the range of the increment distribution.

We can extend this simple construction much further, and we shall do so in Chapter 15 in particular, where we show that the geometric drift condition exhibited by the linear model implies much more, including rates of convergence results, than we have so far described.

11.4.2 The GI/G/1 queue with re-entry

In Section 2.4.2 we described models for GI/G/1 queueing systems. We now indicate one class of models where we generalize the conditions imposed on the arrival stream and service times by allowing re-entry to the system, and still find conditions under which the queue is positive Harris recurrent.

As in Section 2.4.2, we assume that customers enter the queue at successive time instants $0 = T'_0 < T'_1 < T'_2 < T'_3 < \cdots$. Upon arrival, a customer waits in the queue if necessary, and then is serviced and exits the system. In the G1/G/1 queue, the interarrival times $\{T'_{n+1} - T'_n : n \in \mathbb{Z}_+\}$ and the service times $\{S_i : i \in \mathbb{Z}_+\}$ are i.i.d. and independent of each other with general distributions, and means $1/\lambda$, $1/\mu$ respectively.

After being served, a customer exits the system with probability r and re-enters the queue with probability $1 - r$. Hence the effective rate of customers to the queue is, at least intuitively,

$$\lambda_r := \frac{\lambda}{r}.$$

If we now let N_n denote the queue length (not including the customer which may be in service) at time T'_n-, and this time let R^+_n denote the residual service time (set to zero if the server is free) for the system at time T'_n-, then the stochastic process

$$\Phi_n = \begin{pmatrix} N_n \\ R^+_n \end{pmatrix}, \qquad n \in \mathbb{Z}_+,$$

is a Markov chain with stationary transition probabilities evolving on the ladder-structure space $\mathsf{X} = \mathbb{Z}_+ \times \mathbb{R}_+$.

Now suppose that the load condition

$$\rho_r := \frac{\lambda_r}{\mu} < 1 \tag{11.32}$$

is satisfied. This will be shown to imply positive Harris recurrence for the chain $\boldsymbol{\Phi}$.

Write $[0] = 0 \times 0$ for the state where the queue is empty. Under (11.32), for each $x \in \mathsf{X}$, we may find $m \in \mathbb{Z}_+$ sufficiently large that

$$\mathsf{P}_x\{\Phi_m = [0]\} > 0. \tag{11.33}$$

This follows because under the load constraint, there exists $\delta > 0$ such that with positive probability, each of the first m interarrival times exceeds each of the first m service times by at least δ, and also none of the first m customers re-enter the queue.

For $x, y \in \mathsf{X}$ we say that $x \geq y$ if $x_i \geq y_i$ for $i = 1, 2$. It is easy to see that $\mathsf{P}_x(\Phi_m = [0]) \leq \mathsf{P}_y(\Phi_m = [0])$ whenever $x \geq y$, and hence by (11.33) we have the following result:

Proposition 11.4.3 *Suppose that the load constraint (11.32) is satisfied. Then the Markov chain $\boldsymbol{\Phi}$ is $\delta_{[0]}$-irreducible and aperiodic, and every compact subset of X is petite.* □

We let W_n denote the total amount of time that the server will spend servicing the customers which are in the system at time T'_n+. Let $V(x) = \mathsf{E}_x[W_0]$. It is easily seen that

$$V(x) = \mathsf{E}[W_n \mid \Phi_n = x],$$

and hence that $P^n V(x) = \mathsf{E}_x[W_n]$.

The random variable W_n is also called the *waiting time* of the nth customer to arrive at the queue. The quantity W_0 may be thought of as the total amount of *work* which is initially present in the system. Hence it is natural that $V(x)$, the expected work, should play the role of a Lyapunov function.

The drift condition we will establish for some $k > 0$ is

$$\begin{aligned} &\mathsf{E}_x[W_k] \leq \mathsf{E}_x[W_0] - 1, \qquad x \in A^c \\ &\sup_{x \in A} \mathsf{E}_x[W_k] < \infty; \end{aligned} \tag{11.34}$$

this implies that $V(x)$ satisfies (V2) for the k-skeleton, and hence as in the proof of Theorem 11.1.4 both the k-skeleton and the original chain are regular.

Proposition 11.4.4 *Suppose that $\rho_r < 1$. Then (11.34) is satisfied for some compact set $A \subset \mathsf{X}$ and some $k \in \mathbb{Z}_+$, and hence $\boldsymbol{\Phi}$ is a regular chain.*

Proof Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^2$, and set

$$A_m = \{x \in \mathsf{X} : |x| \leq m\}, \qquad m \in \mathbb{Z}_+.$$

For each $m \in \mathbb{Z}_+$, the set A_m is a compact subset of X.

We first fix k such that $(k/\lambda)(1 - \rho_r) \geq 2$; we can do this since $\rho_r < 1$ by assumption. Let ζ_k then denote the time that the server is active in $[0, T'_k]$. We have

$$W_k = W_0 + \sum_{i=1}^{k} \sum_{j=1}^{n_i} S(i,j) - \zeta_k \tag{11.35}$$

where n_i denotes the number of times that the ith customer visits the system, and the random variables $S(i,j)$ are i.i.d. with mean μ^{-1}.

Now choose m so large that

$$\mathsf{E}_x[\zeta_k] \geq \mathsf{E}_x[T'_k] - 1, \qquad x \in A_m^c.$$

Then by (11.35), and since λ_r/λ is equal to the expected number of times that a customer will re-enter the queue,

$$\begin{aligned} \mathsf{E}_x[W_k] &\leq \mathsf{E}_x[W_0] + \sum_{i=1}^{k} \mathsf{E}_x[n_i](1/\mu) - (\mathsf{E}[T'_k] - 1) \\ &= \mathsf{E}_x[W_0] + (k\lambda_r/\lambda)(1/\mu) - k/\lambda + 1 \\ &= \mathsf{E}_x[W_0] - (k/\lambda)(1-\rho_r) + 1, \end{aligned}$$

and this completes the proof that (11.34) holds. □

11.4.3 Regularity of the scalar SETAR model

Let us conclude this section by analyzing the SETAR models defined in (SETAR1) and (SETAR2) by

$$X_n = \phi(j) + \theta(j)X_{n-1} + W_n(j), \qquad X_{n-1} \in R_j;$$

these were shown in Proposition 6.3.6 to be φ-irreducible T-chains with φ taken as Lebesgue measure μ^{Leb} on $\mathbb{R}$ under these assumptions.

In Proposition 9.5.4 we showed that the SETAR chain is transient in the "exterior" of the parameter space; we now use Theorem 11.3.15 to characterize the behavior of the chain in the "interior" of the space (see Figure B.1). This still leaves the characterization on the boundaries, which will be done below in Section 11.5.2.

Let us call the *interior* of the parameter space that combination of parameters given by

$$\theta(1) < 1,\ \theta(M) < 1,\ \theta(1)\theta(M) < 1 \tag{11.36}$$

$$\theta(1) = 1,\ \theta(M) < 1,\ \phi(1) > 0 \tag{11.37}$$

$$\theta(1) < 1,\ \theta(M) = 1,\ \phi(M) < 0 \tag{11.38}$$

$$\theta(1) = \theta(M) = 1,\ \phi(M) < 0 < \phi(1) \tag{11.39}$$

$$\theta(1) < 0,\ \theta(1)\theta(M) = 1,\ \phi(M) + \theta(M)\phi(1) > 0. \tag{11.40}$$

Proposition 11.4.5 *For the SETAR model satisfying (SETAR1)-(SETAR2), the chain is regular in the interior of the parameter space.*

PROOF To prove regularity for this interior set, we use (V2), and show that when (11.36)-(11.40) hold there is a function V and an interval set $[-R, R]$ satisfying the drift condition

$$\int P(x, dy)V(y) \leq V(x) - 1, \qquad |x| > R. \tag{11.41}$$

First consider the condition (11.36). When this holds it is straightforward to calculate that there must exist positive constants a, b such that

$$1 > \theta(1) > -(b/a),$$

$$1 > \theta(M) > -(a/b).$$

If we now take

$$V(x) = \begin{cases} a\,x & x > 0 \\ b\,|x| & x \leq 0 \end{cases}$$

then it is easy to check that (11.41) holds under (11.36) for all $|x|$ sufficiently large.

To prove regularity under (11.37), use the function

$$V(x) = \begin{cases} \gamma x & x > 0 \\ 2\,[\phi(1)]^{-1}\,|x| & x \le 0 \end{cases}$$

for which (11.41) is again satisfied provided

$$\gamma > 2\,|\theta(M)|\,[\phi(1)]^{-1}$$

for all $|x|$ sufficiently large. The sufficiency of (11.38) follows by symmetry, or directly by choosing the test function

$$V(x) = \begin{cases} \gamma'\,|x| & x \le 0 \\ -2\,[\phi(M)]^{-1}\,x & x > 0 \end{cases}$$

with

$$\gamma' > -2\,|\theta(1)|\,[\phi(M)]^{-1}.$$

In the case (11.39), the chain is driven by the constant terms and we use the test function

$$V(x) = \begin{cases} 2\,[\phi(1)]^{-1}\,|x| & x \le 0 \\ 2\,[|\phi(M)|]^{-1}\,x & x > 0 \end{cases}$$

to give the result.

The region defined by (11.40) is the hardest to analyze. It involves the way in which successive movements of the chain take place, and we reach the result by considering the two-step transition matrix P^2.

Let f_j denote the density of the noise variable $W(j)$. Fix j and $x \in R_j$ and write

$$R(k,j) = \{y : y + \phi(j) + \theta(j)x \in R_k\},$$

$$\zeta(k,x) = -\phi(k) - \theta(k)\phi(j) - \theta(k)\theta(j)x.$$

If we take the linear test function

$$V(x) = \begin{cases} a\,x & x > 0 \\ b\,|x| & x \le 0 \end{cases}$$

(with a, b to be determined below), then we have

$$\begin{aligned} \int P^2(x,dy)V(y) \quad &= \quad \sum_{k=1}^{M} a \int_{\zeta(k,x)}^{\infty} (u - \zeta(k,x))[\int_{R(k,j)} f_k(u - \theta(k)w) f_j(w)dw]du \\ &\quad -b \int_{-\infty}^{\zeta(k,x)} (u - \zeta(k,x))[\int_{R(k,j)} f_k(u - \theta(k)w) f_j(w)dw]du. \end{aligned}$$

It is straightforward to find from this that for some $R > 0$, we have

$$\int P^2(x,dy)V(y) \le -bx - (b/2)(\phi(M) + \theta(M)\phi(1)), x \le -R,$$

$$\int P^2(x,dy)V(y) \le ax + (a/2)(\phi(1) + \theta(1)\phi(M)), x \ge R.$$

But now by assumption $\phi(M) + \theta(M)\phi(1) > 0$, and the complete set of conditions (11.40) also give $\phi(1) + \theta(1)\phi(M) < 0$. By suitable choice of a, b we have that the drift condition (11.41) holds for the two-step chain, and hence this chain is regular. Clearly, this implies that the one step chain is also regular, and we are done. □

11.5 Evaluating non-positivity

11.5.1 A drift criterion for non-positivity

Although criteria for regularity are central to analyzing stability, it is also of value to be able to identify unstable models.

Theorem 11.5.1 *Suppose that the non-negative function V satisfies*

$$\Delta V(x) \geq 0, \qquad x \in C^c; \tag{11.42}$$

and

$$\sup_{x \in \mathsf{X}} \int P(x, dy)|V(x) - V(y)| < \infty. \tag{11.43}$$

Then for any $x_0 \in C^c$ such that

$$V(x_0) > V(x), \qquad \textit{for all } x \in C \tag{11.44}$$

we have $\mathsf{E}_{x_0}[\tau_C] = \infty$.

PROOF The proof uses a technique similar to that used to prove Dynkin's Formula. Suppose by way of contradiction that $\mathsf{E}_{x_0}[\tau_C] < \infty$, and let $V_k = V(\Phi_k)$. Then we have

$$\begin{aligned} V_{\tau_C} &= V_0 + \sum_{k=1}^{\tau_C} (V_k - V_{k-1}) \\ &= V_0 + \sum_{k=1}^{\infty} (V_k - V_{k-1}) \mathbb{1}\{\tau_C \geq k\} \end{aligned}$$

Now from the bound in (11.43) we have for some $B < \infty$

$$\sum_{k=1}^{\infty} \mathsf{E}_{x_0}[|\mathsf{E}[(V_k - V_{k-1}) \mid \mathcal{F}_{k-1}^{\Phi}] \mathbb{1}\{\tau_C \geq k\}|] \leq B \sum_{k=1}^{\infty} \mathsf{P}_{x_0}\{\tau_C \geq k\} = B\mathsf{E}_{x_0}[\tau_C]$$

which is finite. Thus the use of Fubini's Theorem is justified, giving

$$\mathsf{E}_{x_0}[V_{\tau_C}] = V_0(x_0) + \sum_{k=1}^{\infty} \mathsf{E}_{x_0}[\mathsf{E}[(V_k - V_{k-1}) \mid \mathcal{F}_{k-1}^{\Phi}] \mathbb{1}\{\tau_C \geq k\}] \geq V_0(x_0).$$

But by (11.44), $V_{\tau_C} < V_0(x_0)$ with probability one, and this contradiction shows that $\mathsf{E}_{x_0}[\tau_C] = \infty$. □

This gives a criterion for a ψ-irreducible chain to be non-positive. Based on Theorem 11.1.4 we have immediately

Theorem 11.5.2 *Suppose that the chain $\boldsymbol{\Phi}$ is ψ-irreducible and that the non-negative function V satisfies (11.42) and (11.43) where $C \in \mathcal{B}^+(\mathsf{X})$. If the set*

$$C_+^c = \{x \in \mathsf{X} : V(x) > \sup_{y \in C} V(y)\}$$

also lies in $\mathcal{B}^+(\mathsf{X})$ then the chain is non-positive.

In practice, one would set C equal to a sublevel set of the function V so that the condition (11.44) is satisfied automatically for all $x \in C^c$.

It is not the case that this result holds without some auxiliary conditions such as (11.43). For take the state space to be $\mathbb{Z}_+$, and define $P(0,i) = 2^{-i}$ for all $i > 0$; if we now choose $k(i) > 2i$, and let

$$P(i,0) = P(i,k(i)) = 1/2,$$

then the chain is certainly positive Harris, since by direct calculation

$$\mathsf{P}_0(\tau_0 \geq n+1) \leq 2^{-n}.$$

But now if $V(i) = i$ then for all $i > 0$

$$\Delta V(i) = [k(i)/2] - i > 0$$

and in fact we can choose $k(i)$ to give any value of $\Delta V(i)$ we wish.

11.5.2 Applications to random walk and SETAR models

As an immediate application of Theorem 11.5.2 we have

Proposition 11.5.3 *If $\boldsymbol{\Phi}$ is a random walk on a half line with mean increment β then $\boldsymbol{\Phi}$ is regular if and only if*

$$\beta = \int w\, \Gamma(dw) < 0.$$

PROOF In Proposition 11.4.1 the sufficiency of the negative drift condition was established. If

$$\beta = \int w\, \Gamma(dw) \geq 0.$$

then using $V(x) = x$ we have (11.42), and the random walk homogeneity properties ensure that the uniform drift condition (11.43) also holds, giving non-positivity. □

We now give a much more detailed and intricate use of this result to show that the scalar SETAR model is recurrent but not positive on the "margins" of its parameter set, between the regions shown to be positive in Section 11.4.3 and those regions shown to be transient in Section 9.5.2: see Figure B.1-Figure B.3 for the interpretation of the parameter ranges. In terms of the basic SETAR model defined by

$$X_n = \phi(j) + \theta(j)X_{n-1} + W_n(j), \qquad X_{n-1} \in R_j$$

we call the *margins* of the parameter space the regions defined by

$$\theta(1) < 1,\ \theta(M) = 1,\ \phi(M) = 0 \tag{11.45}$$

$$\theta(1) = 1,\ \theta(M) < 1,\ \phi(1) = 0 \tag{11.46}$$

$$\theta(1) = \theta(M) = 1,\ \phi(M) = 0,\ \phi(1) \geq 0 \tag{11.47}$$

$$\theta(1) = \theta(M) = 1,\ \phi(M) < 0,\ \phi(1) = 0 \tag{11.48}$$

$$\theta(1) < 0,\ \theta(1)\theta(M) = 1,\ \phi(M) + \theta(M)\phi(1) = 0. \tag{11.49}$$

We first establish recurrence; then we establish non-positivity. For this group of parameter combinations, we need test functions of the form $V(x) = \log(u + ax)$ where u, a are chosen to give appropriate drift in (V1). To use these we will need the full force of the approximation results in Lemma 8.5.2, Lemma 8.5.3, Lemma 9.4.3, and Lemma 9.4.4, which we previously used in the analysis of random walk, and to analyze this region we will also need to assume (SETAR3): that is, that the variances of the noise distributions for the two end intervals are finite.

Proposition 11.5.4 *For the SETAR model satisfying (SETAR1)-(SETAR3), the chain is recurrent on the margins of the parameter space.*

PROOF We will consider the test function

$$V(x) = \begin{cases} \log(u + ax) & x > R > r_{M-1} \\ \log(v - bx) & x < -R < r_1 \end{cases} \tag{11.50}$$

and $V(x) = 0$ in the region $[-R, R]$, where a, b and R are positive constants and u and v are real numbers to be chosen suitably for the different regions (11.45)-(11.49).

We denote the non-random part of the motion of the chain in the two end regions by

$$k(x) = \phi(M) + \theta(M)x$$

and

$$h(x) = \phi(1) + \theta(1)x.$$

We first prove recurrence when (11.45) or (11.46) holds. The proof is similar in style to that used for random walk in Section 9.5, but we need to ensure that the different behavior in each end of the two end intervals can be handled simultaneously.

Consider first the parameter region $\theta(M) = 1$, $\phi(M) = 0$, and $0 \le \theta(1) < 1$, and choose $a = b = u = v = 1$, with $x > R > r_{M-1}$. Write in this case

$$\begin{aligned} V_1(x) &= \mathsf{E}[\log(u + ak(x) + aW(M))\mathbb{1}_{[k(x)+W(M)>R]}] \\ V_2(x) &= \mathsf{E}[\log(v - bk(x) - bW(M))\mathbb{1}_{[k(x)+W(M)<-R]}] \end{aligned} \tag{11.51}$$

so that

$$\mathsf{E}_x[V(X_1)] = V_1(x) + V_2(x).$$

In order to bound the terms in the expansion of the logarithms in V_1, V_2, we use the further notation

$$\begin{aligned} V_3(x) &= (a/(u + ak(x)))\mathsf{E}[W(M)\mathbb{1}_{[W(M)>R-k(x)]}] \\ V_4(x) &= (a^2/(2(u + ak(x))^2))\mathsf{E}[W^2(M)\mathbb{1}_{[R-k(x)<W(M)<0]}] \\ V_5(x) &= (b/(v - bk(x)))\mathsf{E}[W(M)\mathbb{1}_{[W(M)<-R-k(x)]}]. \end{aligned} \tag{11.52}$$

Since $\mathsf{E}(W^2(M)) < \infty$

$$V_4(x) = (a^2/(2(u + ak(x))^2))\mathsf{E}[W^2(M)\mathbb{1}_{[W(M)<0]}] - o(x^{-2}),$$

and by Lemma 8.5.3 both V_3 and V_5 are also $o(x^{-2})$.

For $x > R, u + ak(x) > 0$, and thus by Lemma 8.5.2,

$$V_1(x) \le \Gamma_M(R - k(x), \infty)\log(u + ak(x)) + V_3(x) - V_4(x),$$

while $v - bk(x) < 0$, and thus by Lemma 9.4.3,

$$V_2(x) \le \Gamma_M(-\infty, -R - k(x))(\log(-v + bk(x)) - 2) - V_5(x).$$

By Lemma 9.4.4(i) we also have that the terms

$$-\Gamma_M(-\infty, R - k(x))\log(u + ak(x)) + \Gamma_M(-\infty, -R - k(x))(\log(-v + bk(x)) - 2)$$

are $o(x^{-2})$. Thus by choosing R large enough

$$\begin{aligned} \mathsf{E}_x[V(X_1)] &\le V(x) - (a^2/(2(u + ak(x))^2))\mathsf{E}[W^2(M)\mathbb{1}_{[W(M)<0]}] + o(x^{-2}) \\ &\le V(x), \qquad x > R. \end{aligned} \tag{11.53}$$

For $x < -R < r_1$ and $\theta(1) = 0$, $\mathsf{E}_x[V(X_1)]$ is a constant and is therefore less than $V(x)$ for large enough R.

For $x < -R < r_1$ and $0 < \theta(1) < 1$, consider

$$\begin{aligned} V_6(x) &= \mathsf{E}[\log(u + ah(x) + aW(1))\mathbb{1}_{[h(x)+W(1)>R]}] \\ V_7(x) &= \mathsf{E}[\log(v - bh(x) - bW(1))\mathbb{1}_{[h(x)+W(1)<-R]}]: \end{aligned} \tag{11.54}$$

we have as before

$$\mathsf{E}_x[V(X_1)] = V_6(x) + V_7(x). \tag{11.55}$$

To handle the expansion of terms in this case we use

$$V_8(x) = (a/(u + ah(x)))\mathsf{E}[W(1)\mathbb{1}_{[W(1)>R-h(x)]}]$$

$$V_9(x) = (b/v - bh(x)))\mathsf{E}[W(1)\mathbb{1}_{[W(1)<-R-h(x)]}]$$

$$V_{10}(x) = (b^2/(2(v - bh(x))^2))\mathsf{E}[W^2(1)\mathbb{1}_{[-R-h(x)>W(1)>0]}].$$

Since $\mathsf{E}[W^2(1)] < \infty$

$$V_{10}(x) = (b^2/(2(v - bh(x))^2))\mathsf{E}[W^2(1)\mathbb{1}_{[W(1)>0]}] - o(x^{-2}),$$

and by Lemma 8.5.3, both $V_8(x)$ and $V_9(x)$ are $o(x^{-2})$.

For $x < -R$, $u + ah(x) < 0$, we have by Lemma 9.4.3(i),

$$V_6(x) \le \Gamma_1(R - h(x), \infty)(\log(-u - ah(x)) - 2) - V_8(x),$$

and $v - bh(x) > 0$, so that by Lemma 8.5.2,

$$V_7(x) \le \Gamma_1(-\infty, -R - h(x))\log(v - bh(x)) - V_9(x) - V_{10}(x).$$

Hence choosing R large enough that $v - bh(x) \le v - bx$, we have from (11.55),

$$\begin{aligned} \Gamma_1(-\infty, -R - h(x))\log(v - bh(x)) &\le \Gamma_1(-\infty, -R - h(x))\log(v - bx) \\ &= V(x) - \Gamma_1(-R - h(x), \infty)\log(v - bx). \end{aligned}$$

By Lemma 9.4.4(ii),

$$\Gamma_1(R - h(x), \infty)(\log(-u - ah(x)) - 2) - \Gamma_1(-R - h(x), \infty)\log(v - bx) \le o(x^{-2}),$$

and thus

$$\begin{aligned} \mathsf{E}_x[V(X_1)] &\le V(x) - (b^2/(2(v-bh(x))^2))\mathsf{E}[W^2(1)\mathbb{1}_{W(1)>0]}] + o(x^{-2}) \\ &\le V(x), \qquad x < -R. \end{aligned} \tag{11.56}$$

Finally consider the region $\theta(M) = 1$, $\phi(M) = 0$, $\theta(1) < 0$, and choose $a = -b\theta(M)$ and $v - u = a\phi(1)$. For $x > R > r_{M-1}$, (11.53) is obtained in a manner similar to the above. For $x < -R < r_1$, we look at

$$V_{11}(x) = (a^2/(2(u+ah(x))^2))\mathsf{E}[W^2(1)\mathbb{1}_{[R-h(x)<W(1)<0]}].$$

By Lemma 9.4.3

$$V_6(x) \le \Gamma_1(R-h(x),\infty)\log(u+ah(x)) + V_8(x) - V_{11}(x),$$

and

$$V_7(x) \le \Gamma_1(-\infty,-R-h(x))(\log(-v+bh(x)) - 2) - V_9(x).$$

From the choice of a, b, u and v,

$$\log(u+ah(x)) = \log(v-bx) = V(x),$$

and thus by Lemma 8.5.3 and Lemma 9.4.4(i) for R large enough

$$\begin{aligned} \mathsf{E}_x[V(X_1)] &\le V(x) - (a^2/(2(u+ah(x))^2))\mathsf{E}[W^2(1)\mathbb{1}_{[W(1)<0]}] + o(x^{-2}) \\ &\le V(x), \qquad x < -R. \end{aligned} \tag{11.57}$$

When (11.46) holds, the recurrence of the SETAR model follows by symmetry from the result in the region (11.45).

(ii) We now consider the region where (11.47) holds: in (11.48) the result will again follow by symmetry.

Choose $a = b = u = v = 1$ in the definition of V. For $x > R > r_{M-1}$, (11.53) holds as before. For $x < -R < r_1$, since $1 - h(x) \le 1 - x$,

$$\Gamma_1(-\infty,-R-h(x))\log(1-h(x)) \le \Gamma_1(-\infty,-R-h(x))\log(1-x).$$

From this, (11.56) is also obtained as before.

(iii) Finally we show that the chain is recurrent if the boundary condition (11.49) holds.

Choose $v - u = b\phi(M) = a\phi(1)$, $b = -a\theta(1) = -a/\theta(M)$. For $x > R > r_{M-1}$, consider

$$V_{12}(x) = (b^2/(2(v-bk(x))^2))\mathsf{E}[W^2(M)\mathbb{1}_{[-R-k(x)>W(M)>0]}].$$

By Lemma 9.4.3 we get both

$$V_1(x) \le \Gamma_M(R-k(x),\infty)(\log(-u-ak(x)) - 2) - V_3(x),$$

$$V_2(x) \le \Gamma_M(-\infty,-R-k(x))\log(v-bk(x)) - V_5(x) - V_{12}(x).$$

From the choice of a, b, u and v

$$\Gamma_M(-\infty,-R-k(x))\log(v-bk(x)) = \log(u+ax) - \Gamma_M(-R-k(x),\infty)\log(u+ax),$$

and thus by Lemma 9.4.4(i) and (iii), for R large enough

$$\begin{aligned} \mathsf{E}_x[V(X_1)]] &\leq V(x) - (b^2/(2(v-bk(x))^2))\mathsf{E}[W^2(M)\mathbb{1}_{[W(M)>0]}] + o(x^{-2}) \\ &\leq V(x), \qquad x > R. \end{aligned} \tag{11.58}$$

For $x < -R < r_1$, since

$$\log(u + ah(x)) = \log(v - bx),$$

(11.57) is obtained similarly.

It is obvious that the above test functions V are norm-like, and hence (V1) holds outside a compact set $[-R, R]$ in each case. Hence we have recurrence from Theorem 9.1.8. □

To complete the classification of the model, we need to prove that in this region the model is not positive recurrent.

Proposition 11.5.5 *For the SETAR model satisfying (SETAR1)-(SETAR3), the chain is non-positive on the margins of the parameter space.*

PROOF We need to show that in the case where

$$\phi(1) < 0, \qquad \phi(1)\phi(M) = 1, \qquad \theta(1)\phi(M) + \theta(M) \leq 0$$

the chain is non-positive. To do this we appeal to the criterion in Section 11.5.1.

As we have $\phi(1)\phi(M) = 1$ we can as before find positive constants a, b such that

$$\phi(1) = -ba^{-1}, \qquad \phi(M) = -ab^{-1}.$$

We will consider the test function

$$V(x) = V_{cd}(x) + \mathbb{1}_{kR}(x) \tag{11.59}$$

where the functions V_{cd} and $\mathbb{1}_{kR}$ are defined for positive c, d, k, R by

$$\mathbb{1}_{kR}(x) = \begin{cases} k & |x| \leq R \\ 0 & |x| > R \end{cases}$$

and

$$V_{cd}(x) = \begin{cases} a\,x + c & x > 0 \\ b\,|x| + d & x \leq 0 \end{cases}.$$

It is immediate that

$$\int P(x, dy)|V(x) - V(y)| \leq a\mathsf{E}[|W_1|] + b\mathsf{E}[|W_M|] + 2(a|\theta(1)| + b|\theta(M)|) + 2|d - c|,$$

whilst V is obviously norm-like.

We now verify that indeed the mean drift of $V(\Phi_n)$ is positive. Now for $x \in R_M$, we have

$$\begin{aligned} \int P(x, dy)V(y) &= \int \Gamma_M(dy - \theta(M) - \phi(M)x)V_{cd}(y) \\ &\quad + \int \Gamma_M(dy - \theta(M) - \phi(M)x)\mathbb{1}_{kR}(y), \end{aligned} \tag{11.60}$$

and the first of these terms can be written as

$$\begin{aligned}\int \Gamma_M(dy - \theta(M) - \phi(M)x)V_{cd}(y) \\ = \int \Gamma_M(dz)[-b(z + \theta(M) + \phi(M)x) + d] \\ + \int_{-\theta(M)-\phi(M)x}^{\infty} \Gamma_M(dz)[(a+b)(z + \theta(M) + \phi(M)x) + c - d]. \quad (11.61)\end{aligned}$$

Using this representation we thus have

$$\begin{aligned}\int P(x,dy)V(y) &= ax + d - b\theta(M) \\ &\quad + \int_0^{\infty} \Gamma_M(dy - \theta(M) - \phi(M)x)[(a+b)y + c - d] \\ &\quad + \int_{-R}^{R} k\Gamma_M(dy - \theta(M) - \phi(M)x). \qquad (11.62)\end{aligned}$$

A similar calculation shows that for $x \in R_1$,

$$\begin{aligned}\int P(x,dy)V(y) &= -bx + c - a\theta(1) \\ &\quad - \int_{-\infty}^{0} \Gamma_1(dy - \theta(1) - \phi(1)x)[(a+b)y + c - d] \\ &\quad + \int_{-R}^{R} k\Gamma_1(dy - \theta(1) - \phi(1)x). \qquad (11.63)\end{aligned}$$

Let us now choose the positive constants c, d to satisfy the constraints

$$a\theta(1) \geq d - c \geq b\theta(M) \qquad (11.64)$$

(which is possible since $\theta(1)\phi(M) + \theta(M) \leq 0$) and k, R sufficiently large that

$$R \geq \max(|\theta(1)|, |\theta(M)|) \qquad (11.65)$$

$$k \geq (a+b)\max(|\theta(1)|, |\theta(M)|). \qquad (11.66)$$

It then follows that for all x with $|x|$ sufficiently large

$$\int P(x,dy)V(y) \geq V(x)$$

and the chain is non-positive from Section 11.5.1. □

11.6 Commentary

For countable space chains, the results of this chapter have been thoroughly explored. The equivalence of positive recurrence and the finiteness of expected return times to each atom is a consequence of Kac's Theorem, and as we saw in Proposition 11.1.1, it is then simple to deduce the regularity of all states. As usual, Feller [76] or Chung [49] or Çinlar [40] provide excellent discussions.

Indeed, so straightforward is this in the countable case that the name "regular chain", or any equivalent term, does not exist as far as we are aware. The real focus on regularity and similar properties of hitting times dates to Isaac [103] and Cogburn

[53]; the latter calls regular sets "strongly uniform". Although many of the properties of regular sets are derived by these authors, proving the actual existence of regular sets for general chains is a surprisingly difficult task. It was not until the development of the Nummelin-Athreya-Ney theory of splitting and embedded regeneration occurred that the general result of Theorem 11.1.4, that positive recurrent chains are "almost" regular chains was shown (see Nummelin [201]).

Chapter 5 of Nummelin [202] contains many of the equivalences between regularity and positivity, and our development owes a lot to his approach. The more general f-regularity condition on which he focuses is central to our Chapter 14: it seems worth considering the probabilistic version here first.

For countable chains, the equivalence of (V2) and positive recurrence was developed by Foster [82], although his proof of sufficiency is far less illuminating than the one we have here. The earliest results of this type on a non-countable space appear to be those in Lamperti [152], and the results for general ψ-irreducible chains were developed by Tweedie [275], [276]. The use of drift criteria for continuous space chains, and the use of Dynkin's Formula in discrete time, seem to appear for the first time in Kalashnikov [115, 117, 118]. The version used here and later was developed in Meyn and Tweedie [178], although it is well known in continuous time for more special models such as diffusions (see Kushner [149] or Khas'minskii [134]).

There are many rediscoveries of mean drift theorems in the literature. For operations research models (V2) is often known as Pakes' Lemma from [212]: interestingly, Pakes' result rediscovers the original form buried in the discussion of Kendall's famous queueing paper [128], where Foster showed that a sufficient condition for positivity of a chain on $\mathbb{Z}_+$ is the existence of a solution to the pair of equations

$$\begin{array}{rcll}\sum P(x,y)V(y) & \leq & V(x)-1, & x \geq N \\ \sum P(x,y)V(y) & < & \infty, & x < N,\end{array}$$

although in [82] he only gives the result for $N = 1$. The general N form was also re-discovered by Moustafa [190], and a form for reducible chains given by Mauldon [164]. An interesting state-dependent variation is given by Malyšhev and Men'šikov [159]; we return to this and give a proof based on Dynkin's Formula in Chapter 19.

The systematic exploitation of the various equivalences between hitting times and mean drifts, together with the representation of π, is new in the way it appears here. In particular, although it is implicit in the work of Tweedie [276] that one can identify sublevel sets of test functions as regular, the current statements are much more comprehensive than those previously available, and generalize easily to give an appealing approach to f-regularity in Chapter 14.

The criteria given here for chains to be non-positive have a shorter history. The fact that drift away from a petite set implies non-positivity provided the increments are bounded in mean appears first in Tweedie [276], with a different and less transparent proof, although a restricted form is in Doob ([68], p 308), and a recent version similar to that we give here has been recently given by Fayolle et al [73]. All proofs we know require bounded mean increments, although there appears to be no reason why weaker constraints may not be as effective.

Related results on the drift condition can be found in Marlin [163], Tweedie [274], Rosberg [226] and Szpankowski [261], and no doubt in many other places: we return to these in Chapter 19.

Applications of the drift conditions are widespread. The first time series application appears to be by Jones [113], and many more have followed. Laslett et al [153] give an overview of the application of the conditions to operations research chains on the real line. The construction of a test function for the GI/G/1 queue given in Section 11.4.2 is taken from Meyn and Down [175] where this forms a first step in a stability analysis of generalized Jackson networks. A test function approach is also used in Sigman [239] and Fayolle et al [73] to obtain stability for queueing networks: the interested reader should also note that in Borovkov [27] the stability question is addressed using other means.

The SETAR analysis we present here is based on a series of papers where the SETAR model is analyzed in increasing detail. The positive recurrence and transience results are essentially in Petruccelli et al [214] and Chan et al [43], and the non-positivity analysis as we give it here is taken from Guo and Petruccelli [92]. The assumption of finite variances in (SETAR3) is again almost certainly redundant, but an exact condition is not obvious.

We have been rather more restricted than we could have been in discussing specific models at this point, since many of the most interesting examples, both in operations research and in state-space and time series models, actually satisfy a stronger version of the drift condition (V2): we discuss these in detail in Chapter 15 and Chapter 16. However, it is not too strong a statement that Foster's Criterion (as (V2) is often known) has been adopted as the tool of choice to classify chains as positive recurrent: for a number of applications of interest we refer the reader to the recent books by Tong [267] on nonlinear models and Asmussen [10] on applied probability models. Variations for two-dimensional chains on the positive quadrant are also widespread: the first of these seems to be due to Kingman [135], and on-going usage is typified by, for example, Fayolle [72].