B

Testing for Stability

B.1 A Glossary of Drift Conditions

In this section we collect together the various "Foster-Lyapunov" or "drift" criteria which we have developed for the testing of various forms of stability described in Section A.

In using each of these drift conditions, one is required to find two chain-related characteristics:

(i) a suitable non-negative "test function" which is always denoted V;

(ii) a suitable "test set" which is always denoted C.

Typically, for well-behaved chains we are able without great difficulty to give conditions showing a set C to be a "test set"; these sets are usually petite, or for T-chains, compact. The choice of V, on the other hand, is an art form and depends strongly on intuition regarding the movement of the chain.

The Recurrence Criterion (V1) The weakest stability condition was introduced on page 194. Its use in general requires the existence of a function V, unbounded off petite sets, or norm-like on topological spaces, and a petite or compact set C, with

$$\Delta V(x) \leq 0, \qquad x \in C^c \tag{8.44}$$

Several theorems show this to be an appropriate condition for various forms of recurrence, including Theorem 8.4.3, Theorem 9.4.1, and Theorem 12.3.3.

The Positivity/Regularity Criterion (V2) The second condition (often known as Foster's Condition) was introduced on page 267. We require for some constant $b < \infty$

$$\Delta V(x) \leq -1 + b\mathbb{1}_C(x), \qquad x \in \mathsf{X}, \tag{11.17}$$

where V is allowed to be an extended real-valued function $V\colon \mathsf{X} \to [0, \infty]$ provided it is finite at some point in X, and C is typically petite or compact. Theorems which show this to be an appropriate condition for various forms of regularity, existence of invariant measures, positive recurrence and ergodicity are Theorem 11.3.4, Theorem 11.3.11, Theorem 11.3.15, Theorem 12.3.4, Theorem 12.4.5 and Theorem 13.0.1.

The f-Positivity/f-Regularity Criterion (V3) The third condition was introduced on page 341. Here again V is an extended real-valued function $V: \mathsf{X} \to [0, \infty]$ finite at some point in X and C is typically petite or compact; and we require for some function $f: \mathsf{X} \to [1, \infty)$, and a constant $b < \infty$,

$$\Delta V(x) \leq -f(x) + b\mathbb{1}_C(x), \qquad x \in \mathsf{X}. \tag{14.16}$$

Various theorems which show this to be an appropriate condition for various forms of f-regularity, existence of f-moments of π and f-ergodicity and even sample path results such as the Central Limit Theorem and the Law of the Iterated Logarithm include Theorem 14.2.3, Theorem 14.2.6, Theorem 14.3.7 and Theorem 17.5.3.

The V-Uniform/V-Geometric Ergodicity Criterion (V4) The strongest stability condition was introduced on page 371. Again V is an extended real-valued function $V: \mathsf{X} \to [1, \infty]$ finite at some point in X, and for constants $\beta > 0$ and $b < \infty$,

$$\Delta V(x) \leq -\beta V(x) + b\mathbb{1}_C(x), \quad x \in \mathsf{X}. \tag{15.28}$$

Critical theorems which show this to be an appropriate condition for various forms of V-geometric regularity, geometric ergodicity, V-uniform ergodicity are Theorem 15.2.6 and Theorem 16.1.2. We also showed in Lemma 15.2.8 that (V4) holds with a petite set C if and only if V is unbounded off petite sets and

$$PV \leq \lambda V + L \tag{15.35}$$

holds for some $\lambda < 1$, $L < \infty$, and this is a frequently used alternative form.

The Transience/Nullity Criterion Finally, we introduced conditions for instability. These involve the relation

$$\Delta V(x) \geq 0, \qquad x \in C^c \tag{8.43}$$

which was introduced on both page 281 and page 194.

Theorems which show this to be an appropriate condition for various forms of non-positivity or nullity include Theorem 11.5.1: typically these require V to have bounded increments in expectation, and C to be a sublevel set of V.

Exactly the same drift criterion can also be shown to give an appropriate condition for various forms of transience, as in Theorem 8.4.2: these require, typically, that V be bounded, and C be a sublevel set of V with both C and C^c in $\mathcal{B}^+(\mathsf{X})$.

These criteria form the basis for classification of the chains we have considered into the various stability classes, and despite their simplicity they appear to work well across a great range of cases. It is our experience that in the use of the two commonest criteria (V2) and (V4) for models on $\mathbb{R}^k$, quadratic forms are the most useful to use, although the choice of a suitable form is not always trivial.

Finally, we mention that in some cases where identifying the test function is difficult we may need greater subtlety: the generalizations in Chapter 19 then provide a number of other methods of approach.

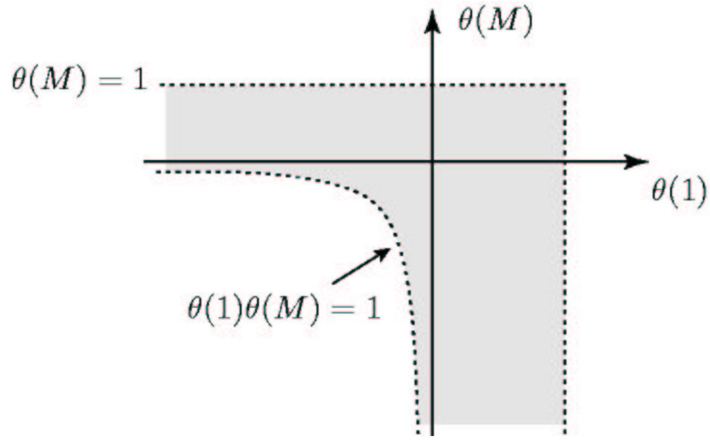

Fig. B.1. The SETAR model: stability classification of $(\theta(1), \theta(M))$-space. The model is regular in the shaded "interior" area (11.36), and transient in the unshaded "exterior" (9.48), (9.49) and (9.52). The boundaries are in the figures below.

B.2 The scalar SETAR Model: a complete classification

In this section we summarize, for illustration, the use of these drift conditions in practice for scalar first order SETAR models: recall that these are piecewise linear models satisfying

$$X_n = \phi(j) + \theta(j)X_{n-1} + W_n(j), \qquad X_{n-1} \in R_j$$

where $-\infty = r_0 < r_1 < \cdots < r_M = \infty$ and $R_j = (r_{j-1}, r_j]$; for each j, the noise variables $\{W_n(j)\}$ form an i.i.d. zero-mean sequence independent of $\{W_n(i)\}$ for $i \neq j$.

We assume (for convenience of exposition) that the following conditions hold on the noise distributions:

(i) each $\{W_n(i)\}$ has a density positive on the whole real line, and

(ii) the variances of the noise distributions for the two end intervals are finite.

Neither of these conditions is necessary for what follows, although weakening them makes proofs rather more difficult.

In Figure B.1, Figure B.2 and Figure B.3 we depict the parameter space in terms of $\phi(1), \theta(1), \phi(M)$, and $\theta(M)$. The results we have proved show that in the "interior" and "boundary" areas, the SETAR model is Harris recurrent; and it is transient in the "exterior" of the parameter space. In accordance with intuition, the model is null on the boundaries themselves, and regular (and indeed, in this case, geometrically ergodic) in the strict interior of the parameter space.

The steps taken to carry out this classification form a template for analyzing many models, which is our reason for reproducing them in summary form here.

(STEP 1) As a first step, we show in Theorem 6.3.6 that the SETAR model is a φ-irreducible T-process with φ taken as Lebesgue measure μ^{Leb} on $\mathbb{R}$. Thus compact sets are test sets in all of the criteria above.

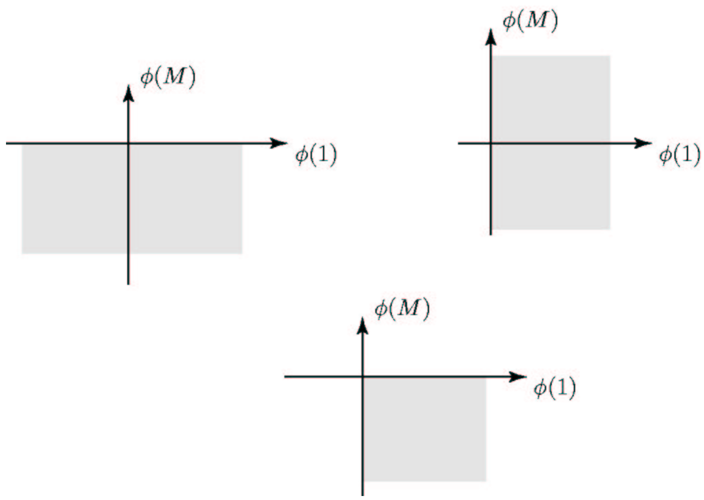

Fig. B.2. The SETAR model: stability classification of $(\phi(1), \phi(M))$-space in the regions $(\theta(M) = 1; \theta(1) \leq 1)$ and $(\theta(M) \leq 1; \theta(1) = 1)$. The model is regular in the shaded "interior" areas, which are clockwise (11.38), (11.37) and (11.39); transient in the unshaded "exterior" (9.51), (9.50); and null recurrent on the "margins" described clockwise by (11.45), (11.46) and (11.47)–(11.48).

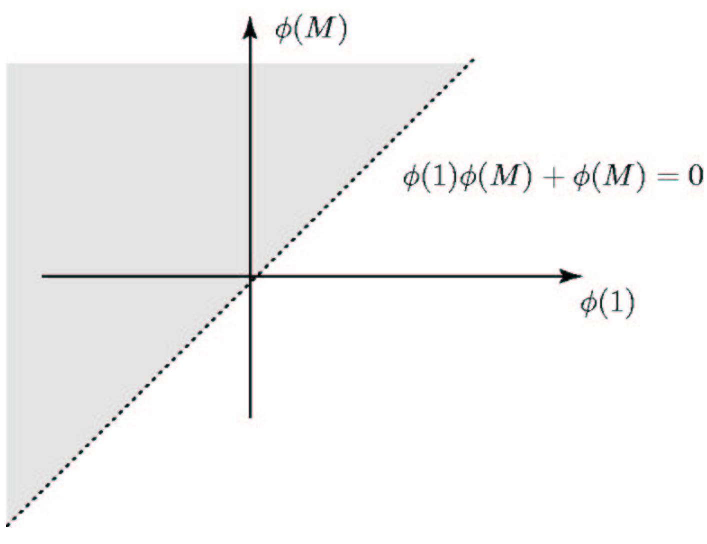

Fig. B.3. The SETAR model: stability classification of $(\phi(1), \phi(M))$-space in the region $(\theta(M)\,\theta(1) = 1; \theta(1) \leq 0)$. The model is regular in the shaded "interior" area (11.40); transient in the unshaded "exterior" (9.53); and null recurrent on the "margin" described by (11.49).

(STEP 2) In the "interior" of the parameter space we are able to identify geometric ergodicity in Proposition 11.4.5, by using (V4) with linear test functions of the form

$$V(x) = \begin{cases} a\,x & x > 0 \\ b\,|x| & x \leq 0 \end{cases}$$

and suitable choice of the coefficients a, b, related to the parameters of the model. Note that we only indicated that V satisfied (V2), but the stronger form is actually proved in that result.

(STEP 3) We establish transience on the "exterior" of the parameter space as in Proposition 9.5.4 using the bounded function

$$V(x) = \begin{cases} 1 - 1/a(x+u), & x > c/a - u \\ 1 - 1/c & -c/b - v < x < c/a - u \\ 1 + 1/b(x+v) & x < -c/b - v \end{cases}$$

for suitable u, v, a, b, c: this satisfies (8.43) so that Theorem 8.4.2 applies.

(STEP 4) Null recurrence is, as is often the case, the hardest to establish. Firstly, Proposition 11.5.4 shows the chain to be recurrent on the boundaries of the parameter space. This is done by applying (V1) with a logarithmic test function

$$V(x) = \begin{cases} \log(u + ax) & x > R > r_{M-1} \\ \log(v - bx) & x < -R < r_1 \end{cases}$$

and $V(x) = 0$ in the region $[-R, R]$, where a, b, R, u and v are constants chosen suitably for different regions of the parameter space.

To complete the classification of the model, we need to prove that in this region the model is not positive recurrent. In Proposition 11.5.5 we show that the chain is indeed null on the margins of the parameter space, using essentially linear test functions in (11.42).

This model, although not linear, is sufficiently so that the methods applied to the random walk or the simple autoregressive models work here also. In this sense the SETAR model is an example of greater complexity but not of a step-change in type. Indeed, the fact that the drift conditions only have to hold outside a compact set means that for this model we really only have to consider the two linear models one each of the end intervals, rendering its analysis even more straightforward.

For more detail on this model see Tong [267]; and for some of the complications in moving to multidimensional versions see Brockwell, Liu and Tweedie [33].

Other generalized random coefficient models or completely nonlinear models with which we have dealt are in many ways more difficult to classify. Nevertheless, steps similar to those above are frequently the only ones available, and in practice linearization to enable use of test functions of these forms will often be the approach taken.